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ON H-EQUIVALENCE OF UNIFORMITIES (THE ISBELL-SMITH PROBLEM)

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I have recently given an example of two different uniformities for the same set X, such that the corresponding Hausdorff uniformities for the set of nonempty subsets of X are topologically equivalent; when this is the case we shall call the original uniformities H-equivalent. The problem posed by Isbell and discussed in a recent paper by D. H. Smith may therefore be reformulated as follows:- (a) Under what conditions are two uniformities H-equivalent? (b) Under what conditions does H-equivalence of uniformities imply identity? The theorems given below supply an answer to (a) and a partial answer to (b). In particular, they show that neither R^n nor Q^n (Q denoting the set of rational numbers with the usual metric) has any other uniformity H-equivalent to its metric uniformity. In a sense, therefore, the example in (1) is the simplest possible one of its kind, though we give in the course of this paper another simple example using transfinite ordinals.

TERMINOLOGY. Let $\mathfrak{U}, \mathfrak{V}$ be two uniformities for the same set X, and let $X_1 \subset X_2 \subset X$. We say that $\mathfrak{U}$ is *uniformly finer than* $\mathfrak{V}$ *on* X_1 *over* X_2 if and only if, given any $V \in \mathfrak{V}, \exists U \in \mathfrak{U}$ such that $U \cap (X_1 \times X_2) \subset V$; usually we take $X_2 = X$. (The use of the different words 'on' and 'over', and the, logically unnecessary, condition $X_1 \subset X_2$, are intended to suggest the motivation and use of the definition.) We say also (a) that $\mathfrak{U}$ is *proximity-finer than* $\mathfrak{V}$ if and only if every pair of sets A, B which are $\mathfrak{V}$-remote (i.e. such that $V(A) \cap B = \phi$ for some $V \in \mathfrak{V}$) are also $\mathfrak{U}$-remote; (b) that $\mathfrak{U}$ is *H-finer than* $\mathfrak{V}$ if and only if the topology of its Hausdorff uniformity $\mathbf{\mathfrak{U}}$ is finer than that of the Hausdorff uniformity $\mathbf{\mathfrak{V}}$ corresponding to $\mathfrak{V}$; i.e. if and only if given any (nonempty) $E_0 \subset X$ and any $V \in \mathfrak{V}, \exists U \in \mathfrak{U}$ (depending on E_0) such that $E \subset U(E_0)$ and $E_0 \subset U(E)$ together imply $E \subset V(E_0)$ and $E_0 \subset V(E)$[1]. The corresponding phrases with 'coarser than' or 'equivalent to' are defined similarly. Note that we use 'finer' in the wide sense, allowing possible equivalence; also that in discussing subsets of X we shall frequently omit the word 'nonempty' where it is obviously implied. Finally, we say (cf. (1)) that a set E is *V-discrete* ($V \in \mathfrak{V}$) if and only if, for x and y in E, $(x, y) \in V$ implies $x = y$, and $\mathfrak{V}$-*discrete*

[1] While this is the form in which the definition, derived from that of the Hausdorff uniformity, is naturally phrased, it is easily seen that the implications are actually respective rather than joint.