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# ON THE BIHARMONIC WAVE EQUATION

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**Under appropriate restrictions of material and motion the equation of motion for a vibrating elastic bar is $(\partial_x^4 + \partial_t^2)u = 0$. Because of its mechanical importance, there is a large literature devoted to the eigenvalue problem for this equation but solutions of boundary value problems for the equation itself seem to have been ignored. It appears that Pini was the first to seek a solution in terms of integrals analogous to thermal potentials. Like Pini, we use a fundamental solution very similar to that of the heat kernel to obtain potential terms which lead to a system of integral equations. While Pini uses Laplace transforms to obtain solutions to the integral equations, we observe that the problem may be reduced to one integral equation of a complex valued function, $f = a + \lambda k * \bar{f}$, effecting a significant simplification.**

**Along the way, we obtain, by reduction to Abel integral equations, a general method of solving semi-infinite problems which can solve boundary value problems not available to Fourier transforms, the technique presently used.**

The first appendix is a justification of the change of order of integration for a key iterated integral; the computation of some important integrals is given in the second appendix.

**2. The fundamental solutions.** By standard Fourier transform techniques, one finds that a fundamental solution for the equation

$$(\partial_x^4 + \partial_t^2)u = 0 \tag{1}$$

is

$$K(x, t) = -\pi^{-1/2}t^{1-1/2} \exp\left(\frac{ix^2}{4t} + i\frac{\pi}{4}\right). \tag{2}$$

We also define

$$C(x, t) = \operatorname{Re} K(x, t), \quad S(x, t) = \operatorname{Im} K(x, t). \tag{3}$$

We obtain by straightforward computation:

$$\partial_x K = \frac{ix}{2t} K, \tag{4}$$

$$(\partial_x^2 + i\partial_t)K = 0, \tag{5}$$

$$(\partial_x^2 - i\partial_t)\bar{K} = 0, \tag{6}$$