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THE δ^2-PROCESS AND RELATED TOPICS

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This paper deals with (1) acceleration of the convergence of a convergent complex series, (2) rapidity of convergence, and (3) sufficient criteria for the divergence of a complex series. Various results of Samuel Lubkin, Imanuel Marx and J. P. King which concern or are closely related to Aitkin's δ^2-process are generalized. Some typical results are as follows:

(1) If a complex series and its δ^2-transform converge, their sums are equal.

(2) Suppose that $\Sigma a_n, \Sigma b_n$ are complex series such that $b_n/a_n \to 0$, and A, B exists such that $|a_n/a_{n-1}| \leqq A < 1/2$, $|b_n/b_{n-1}| \leqq B < 1$ for all sufficiently large n. Then Σb_n converges more rapidly than Σa_n.

(3) If the sequence $\{1/a_n - 1/a_{n-1}\}$ is bounded, then the complex series Σa_n diverges.

Given a convergent complex series $\Sigma a_n = S$, quantities $T_n = (a_n + a_{n+1} + \cdots)/a_{n-1}$ are used to obtain results on accelerating the convergence of Σa_n and on rapidity of convergence. The convergence of $\{T_n\}$ is treated and corresponding necessary and sufficient conditions are established for the transform $\Sigma a_{\alpha n} = S$ to converge more rapidly that Σa_n, where $a_{\alpha 0} = a_0 + a_1\alpha_1$, $a_{\alpha n} = a_n + a_{n+1}\alpha_{n+1} - a_n\alpha_n$ for $n \geqq 1$, and $\{\alpha_n\}$ is any complex sequence. Divergence theorems are proven, of which Theorem 2.8 furnishes a generalization of corrected results of Marx [10] and King [7]. The appropriate corrections are indicated in Tucker [16]. These divergence theorems are used to prove that if Σa_n and its δ^2-transform are convergent complex series, their sums are equal. This fact was first published by Lubkin [9] for real series. Theorem 2.9 gives a generalization of a theorem of Marx [10] and King [7], corrected statements of which are given in Tucker [16]. Some related theorems on rapidity of convergence are then proven. Before turning to the general analysis, we now present difinitions, notations and certain elementary facts relevant to acceleration.

Given a complex series $\sum_0^\infty a_n$, we shall write Σa_n for $\sum_0^\infty a_n$, $S_n = \sum_0^n a_k$, and, if Σa_n converges, $S = \Sigma a_n$. Similarly, if $\Sigma a'_n$ converges, then $S' = \Sigma a'_n$. Given two convergent series Σa_n and $\Sigma a'_n$, the latter is said to converge more rapidly than the former if and only if $(S' - S'_n)/(S - S_n) \to 0$ as $n \to \infty$. If Σa_n converges, "$MR(\Sigma a_n)$" will denote the class of all series Σb_n which converge more rapidly to S than Σa_n.

The concept of "acceleration" or "speed-up" can now be defined as the problem of finding a series Σb_n such that $\Sigma b_n \in MR(\Sigma a_n)$. We