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SOME TOPOLOGICAL PROPERTIES OF PIERCING POINTS

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Let K be the closure of one of the complementary domains of a 2-sphere S topologically embedded in the 3-sphere, S^3. We give first (Theorem 1) a characterization of those points $p \in S$ with the following property: there exists a homeomorphism $h: K \to S^3$ such that $h(S)$ can be pierced with a tame arc at $h(p)$. The topological property of K which distinguishes such a "piercing point" p is this: $K - p$ is 1-ULC. Using this result, we find (Theorems 2 and 3) that p is a piercing point of K if and only if S is arcwise accessible at p by a tame arc from $S^3 - K$ (note: perhaps S cannot be pierced with a tame arc at p, even if p is a piercing point of K). Thus, the "tamely arcwise accessible" property is independent of the embedding of K in S^3. The corollary to Theorem 2 gives an alternate proof of an as yet unpublished fact, first proven by R. H. Bing: a topological 2-sphere in S^3 is arcwise accessible at each point by a tame arc from at least one of its complementary domains.

In the last section of the paper, we give two applications of the above theorems. First, we show in Theorem 4 that S can be pierced with a tame arc at p if and only if p is a piercing point of both K and the closure of $S^3 - K$. Finally, we remark in Theorem 5 that S can be pierced with a tame arc at each of its points if it is "free" in the sense that for each $\varepsilon > 0$, S can be mapped into each of its complementary domains by a mapping which moves each point less than ε. It is not known whether each 2-sphere S with this last property is tame.

A space homeomorphic to such a set K in S^3 (as described at the beginning of the Introduction) is called a *crumpled cube*. We write Bd $K = S$ and Int $K = K -$ Bd K. An arc A in S^3 is said to *pierce* a 2-sphere S in S^3 if $A \cap S$ is an interior point p of A and the two components of $A - p$ lie in different components of $S^3 - S$. The *piercing points of a crumpled cube* are defined as above and were first considered by Martin [10]. It follows from Lemmas 2 and 3 of [10] and [6; Th. 11] that the nonpiercing points of a crumpled cube K form a 0-dimensional F_σ subset of Bd K.

If C and D are subsets of a space Y with metric d, and $\varepsilon > 0$, we use $B(C, D; \varepsilon)$ to denote the set of all points $x \in D$ such that for some $y \in C$, $d(x, y) < \varepsilon$. The metric on E^3 and S^3 is always assumed to be the ordinary Euclidean one. Let $\Delta^n (n \geqq 1)$ denote a closed n-