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ON THE SQUARE-FREENESS OF FERMAT AND MERSENNE NUMBERS

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It has been conjectured that the Fermat and Mersenne numbers are all square-free. In this note it is shown that if some Fermat or Mersenne number fails to be square-free, then for any prime p whose square divides the appropriate number, it must be that $2^{p-1} \equiv 1 \pmod{p^2}$. At present there are only two primes known which satisfy the above congruence. It is shown that neither of these two primes is a factor of any Fermat or Mersenne number.

Those odd primes p for which $2^{p-1} \equiv 1 \pmod{p^2}$ have long been of interest. No doubt much of this interest has been generated by Wieferich's theorem, which states that if Fermat's equation $x^p + y^p + z^p = 0$ has a solution in integers with p an odd prime and $xyz \not\equiv 0 \pmod{p}$, then $2^{p-1} \equiv 1 \pmod{p^2}$.

Throughout, "p" and "q" will denote odd primes; "n" is a positive integer other than 1; "$2Rp$" indicates that 2 is a quadratic residue modulo p; "$o(2, p)$" is the exponent to which 2 belongs modulo p; and $F_n = 2^{2^n} + 1$ and $M_q = 2^q - 1$.

Our result follows immediately from the following theorem which proves a bit more than has been indicated so far.

THEOREM 1. *If p divides some F_n [some M_q], then* $2^{(p-1)/2} \equiv 1 \pmod{F_n}$ $[2^{(p-1)/2} \equiv 1 \pmod{M_q}]$.

Proof. Let $p \mid F_n$, then $2^{2^n} \equiv -1 \pmod{p}$ and $2^{2^{n+1}} \equiv 1 \pmod{p}$ so that $o(2, p) \mid 2^{n+1}$ and $o(2, p) \nmid 2^n$. It follows that $o(2, p) = 2^{n+1}$. Now $2^{p-1} \equiv 1 \pmod{p}$ which implies that $2^{n+1} \mid (p-1)$ and

$$p \equiv 1 \pmod{8} . \tag{1}$$

Hence $2Rp$ and by Euler's criterion $2^{(p-1)/2} \equiv 1 \pmod{p}$ so that $2^{n+1} \mid ((p-1)/2)$. It follows that $(2^{2^{n+1}} - 1) \mid (2^{(p-1)/2} - 1)$. Clearly $F_n \mid (2^{2^{n+1}} - 1)$, and therefore $F_n \mid (2^{(p-1)/2} - 1)$.

Let $p \mid M_q$, then $2^q \equiv 1 \pmod{p}$ and $2^{q+1} \equiv 2 \pmod{p}$. Since $q + 1$ is even, we obtain that $2Rp$ and therefore

$$p \equiv \pm 1 \pmod{8} . \tag{2}$$

Also $o(2, p) \mid q$ so that $o(2, p) = q$. As before we get that