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UNITARY OPERATORS IN BANACH SPACES

T. V. PANCHAPAGESAN

The notion of hermitian operators in Hilbert space has been extended to Banach spaces by Lumer and Vidav. Recently, Berkson has shown that a scalar type operator S in a Banach space X can be decomposed into $S = R + iJ$ where (i) R and J commute and (ii) $R^m J^n (m, n = 0, 1, 2, \cdots)$ are hermitian in some equivalent norm on X. The converse is also valid if the Banach space is reflexive. Thus we see that the scalar type operators in a Banach space play a role analogous to the normal operators in a Hilbert space.

In this paper, the well-known Hilbert space notion of unitary operators is suitably extended to operators in Banach spaces and a polar decomposition is obtained for a scalar type operator. It is further shown that this polar decomposition is unique and characterises scalar type operators in reflexive Banach spaces. Finally, an extension of Stone's theorem on one-parameter group of unitary operators in Hilbert spaces is obtained (under suitable conditions) for reflexive Banach spaces.

The terminology and notation in this paper are as follows. The term Banach space always means a complex Banach space. The Banach algebra of all operators on a Banach space X is denoted by $B(X)$. For an operator T on a Banach space X, the spectrum of T is denoted by $\sigma(T)$. The term spectral operator on a Banach space X refers to a spectral operator of class X^*. The exponential function at the operator T is defined and denoted as below.

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!} .$$

DEFINITION 1. An operator T on a Banach space X is said to be hermitian under the equivalent norm $||\cdot||$ on X, if $[Tx, x]$ is real for all x in X with $||x|| = 1$, where $[\ ,\ ]$ is some semi-inner-product on X, inducing the norm $||\cdot||$.

The above definition is due to Lumer and in [7] it is shown that

(1.1) an operator T is hermitian under the equivalent norm $||\cdot||$ on X if and only if $||I + irT|| = 1 + o(r)$ for real r, as r tends to zero.

Further as a consequence of Lemma 1 of Vidav [9], it follows that

(1.2) if T is hermitian under the equivalent norm $||\cdot||$ on X, then $||e^{irT}|| = 1$ for all real r.