PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 1, 1967

# ON THE RELATIONSHIP BETWEEN HAUSDORFF DIMENSION AND METRIC DIMENSION

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**The definitions of the Hausdorff dimension $\dim_h X$, upper metric dimension $\overline{\dim}\, X$ and lower metric dimension $\underline{\dim}\, X$ of a metric space $X$ all depend upon asymptotic characteristics of diameters of sets in covers of $X$. We relate these notions. First we note that $\dim_h X \leq \underline{\dim}\, X$ holds for all totally bounded metric spaces $X$, while on the other hand there exist perfect subsets $A$ of $[0, 1]$ satisfying $\dim_h A = 0$ and $\underline{\dim}\, A = 1 = \underline{\dim}\,[0,1]$. Finally we show that there exist perfect subsets $S$ of $[0,1]$ which satisfy $\dim_h S = 0$ and $\overline{\dim}\, S = 1$ even when strong local conditions are imposed.**

The notions of Hausdorff dimension (see 1, 2) and metric dimension (see 5 p. 296, 8) are closely related; in fact most compact metric spaces encountered in analysis have the same Hausdorff and metric dimensions. In this paper we investigate some aspects of the relationship between these two concepts.

By the Hausdorff dimension of a subset $E$ of a metric space is meant the number $\dim_h E = \sup\{p\colon \mu_p^*(E) = +\infty\}$, where $\mu_p^*(E)$ is defined to be $+\infty$ if $p = 0$ and $\mu_p^*(E) = \sup_{\varepsilon>0} l(E, p; \varepsilon)$ if $p > 0$,

$$(1) \qquad l(E, p; \varepsilon) = \inf\left\{\sum_{i=1}^{+\infty} (\operatorname{diam} E_i)^p\colon E \subset \bigcup_{i=1}^{+\infty} E_i,\ \operatorname{diam} E_i \leqq \varepsilon \text{ for each } i = 1, 2, \cdots\right\}.$$

For each totally bounded subset $A$ of a metric space (i.e. each subset which for each $\varepsilon > 0$ can be covered by a finite number of sets of diameter not exceeding $\varepsilon$) the upper metric dimension $\overline{\dim}\, A$ and lower metric dimension $\underline{\dim}\, A$ of $A$ are defined as follows (all logarithms have base 2):

$$(2) \qquad \overline{\dim}\, A = \overline{\lim_{\varepsilon\to 0+}} (\log N_\varepsilon(A))/\log(\varepsilon^{-1})$$

and

$$(3) \qquad \underline{\dim}\, A = \underline{\lim}_{\varepsilon\to 0+} (\log N_\varepsilon(A))/\log(\varepsilon^{-1}) ,$$

where, for each $\varepsilon > 0$, $N_\varepsilon(A)$ denotes the smallest number of sets in any cover of $A$ by sets of diameter not exceeding $2\varepsilon$. It is customary (see 5, p. 280) to abbreviate $\log N_\varepsilon(A)$ by $H_\varepsilon(A)$; this function has