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LINEAR TRANSFORMATIONS WHICH PRESERVE HERMITIAN AND POSITIVE SEMIDEFINITE OPERATORS

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Let $\mathfrak{A}$ and $\mathfrak{B}$ represent the full algebras of linear operators on the finite-dimensional unitary spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. The symbol $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ will denote the complex space of all linear maps from $\mathfrak{A}$ to $\mathfrak{B}$. This paper concerns itself with the study of the following two cones in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$:

(i) the cone $\mathscr{C}$ of all $T \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ which send hermitian operators in $\mathfrak{A}$ to hermitian operators in $\mathfrak{B}$, and

(ii) the subcone $\mathscr{C}^+$ (of $\mathscr{C}$) of all $T \in \mathscr{L}(\mathfrak{A}, \mathfrak{B})$ which send positive semidefinite operators in $\mathfrak{A}$ to positive semidefinite operators in $\mathfrak{B}$.

In our main results, we characterize the transformations in the cone $\mathscr{C}$ (Theorem 2.1) and present a structure theorem concerning the transformations in the cone $\mathscr{C}^+$ (Theorem 2.3). Identifying operators in the algebras $\mathfrak{A}$ and $\mathfrak{B}$ with appropriate square matrices, we may summarize Theorem 2.1 by saying that any and every linear transformation $\boldsymbol{T}$ which preserves hermitian matrices is of the form $\boldsymbol{T}: A \rightarrow \sum \alpha_i X_i^* A^t X_i$, where each α_i is a real scaler, and each X_i is a certain rectangular matrix depending on $\boldsymbol{T}$; X_i^* and A^t represent the conjugate transpose and the transpose of matrices X_i and A, respectively. Theorem 2.3 says that the cone of positive semidefinite-preserving transformations $\mathscr{C}^+$ "generates" or spans all of $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ in the sense that any $\boldsymbol{T}$ in $\mathscr{L}(\mathfrak{A}, \mathfrak{B})$ can be written

$$\boldsymbol{T} = (\boldsymbol{K}_1 - \boldsymbol{K}_2) + i(\boldsymbol{K}_3 - \boldsymbol{K}_4) ,$$

where $i^2 = -1$, and each $\boldsymbol{K}_i$ is an element of $\mathscr{C}^+$.

1. Preliminaries.

$L(\mathscr{K}, \mathscr{H})$ denotes the space of linear transformations from the Hilbert space $\mathscr{K}$ to the Hilbert space $\mathscr{H}$. We define:

1 (a). $(x \times y)$—the dyad transformation, an element of $L(\mathscr{K}, \mathscr{H})$, is defined for fixed $x \in \mathscr{H}$ and $y \in \mathscr{K}$ by: $(x \times y)(z) = (z, y)x$ for all $z \in \mathscr{K}$, where (z, y) is the inner product of z with y. As it turns out, $(x, y) = \operatorname{tr}((x \times y))$, the trace of $(x \times y)$. If $A \in \mathfrak{A}(=(L(\mathscr{H}, \mathscr{H}))$ and $B \in \mathfrak{B}(=L(\mathscr{K}, \mathscr{K}))$, then $(A(x) \times B(y)) = A(x \times y)B^*$.

1 (b). P_x—denotes the orthogonal projection onto the subspace spanned by x, i.e., for $(x, x) = 1$, we have $P_x = (x \times x)$.