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# ON $w^*$-SEQUENTIAL CONVERGENCE AND QUASI-REFLEXIVITY

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**This paper characterizes quasi-reflexive Banach spaces in terms of certain properties of the $w^*$-sequential closure of subspaces. A real Banach space $X$ is quasi-reflexive of order $n$, where $n$ is a nonnegative integer, if and only if the canonical image $J_X X$ of $X$ has algebraic codimension $n$ in the second dual space $X^{**}$. The space $X$ will be said to have property $P_n$ if and only if every norm-closed subspace $S$ of $X^*$ has codimension $\leqq n$ in its $w^*$-sequential closure $K_X(S)$. By use of a theorem of Singer it is proved that $X$ is quasi-reflexive of order $\leqq n$ if and only if every norm-closed separable subspace of $X$ has property $P_n$. A certain parameter $Q^{(n)}(X)$ is shown to have value 1 if $X$ has property $P_n$ and to be infinite if $X$ does not have $P_n$. The space $X$ has $P_0$ if and only if $w$-sequential convergence and $w^*$-sequential convergence coincide in $X^*$. These results generalize a theorem of Fleming, Retherford, and the author.**

2. If $X$ is a real Banach space, $S$ a subspace of $X^*$, and $K_X(S)$ the $w^*$-sequential closure of $S$ in $X^*$, then $K_X(S)$ is a Banach space under the norm $\varphi_S$ defined by

$$\varphi_S(f) = \inf\left\{\sup_{n\in\omega} \|f_n\| : \{f_n\} \subset S, f_n \xrightarrow{w^*} f\right\}$$

for $f \in K_X(S)$ [5]. If $S \subseteqq T \subseteqq K_X(S)$, let

$$C_X(S, T) = \sup\{\varphi_S(f): f \in T, \|f\| \leqq 1\} .$$

Thus, $K_X(S)$ is norm-closed in $(X^*, \|\ \|)$ if and only if $C_X(S, K_X(S))$ is finite [5]. For each integer $n \geqq 0$ let $\mathscr{T}_n(S)$ be the family of all subspaces $T$ of $X^*$ such that $S \subseteqq T \subseteqq K_X(S)$ and such that $K_X(S)$ is the algebraic direct sum of $T$ and a subspace of dimension $\leqq n$. Let

$$C_X^{(n)}(S) = \inf\{C_X(S, T) : T \in \mathscr{T}_n(S)\} ,$$

and let

$$Q^{(n)}(X) = \sup\{C_X^{(n)}(S) : S \text{ a subspace of } X^*\} .$$

It will be said that $X$ has *property* $P_n$ if and only if $S \in \mathscr{T}_n(S)$ for every norm-closed subspace $S$ of $(X^*, \|\ \|)$.

3. THEOREM 1. *Let $X$ be a real Banach space and $n$ a non-*