PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 3, 1967

A NOTE ON SEMI-PRIMARY HEREDITARY RINGS

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We give an example of two nonisomorphic semi-primary hereditary rings, Ω and Σ with radicals M and N' respectively, such that $\Omega/M^2 = \Sigma/N'^2$.

Let Λ be a semi-primary ring i.e. its (Jacobson) radical N is nilpotent and $\Gamma = \Lambda/N$ is an Artinian ring. The problem of characterizing a semi-primary ring Λ all of whose residue rings have finite global dimension—was dealt in several papers. It turns out that Λ is such a ring if and only if Λ is a residue ring of a semi-primary hereditary ring Ω. It was suggested that Ω is uniquely determined up to an isomorphism by the condition $\Omega/M^2 \approx \Lambda/N^2$, where M is the radical of Ω.

One can prove that if Λ is an epimorphic image of a semi-primary hereditary ring Ω, then Ω is uniquely determined (up to an isomorphism) by the conditions (a) Ω admits a (semi direct sum) splitting, $\Omega = \Gamma + A + M^2$ and (b) $\Omega/M^2 \approx \Lambda/N^2$.

The following ring furnish a counter example to the uniqueness statement if we don't assume condition (a), even if Λ admits a splitting.

Let k be a field of characteristic $p \neq 0$, and let x be a transcendental element over k. Let $R = k(x^{1/p}) \otimes_{k(x)} k(x^{1/p})$ and let V be the radical of R. Then V contains the nonzero element $x^{1/p} \otimes 1 - 1 \otimes x^{1/p}$. Let Σ be a subring of the 3×3 matrix algebra over R, which consists of all matrices M for which:

$$\begin{array}{lll} M_{11} \in k(x^{1/p}) \otimes_{k(x)} 1 & M_{12} = 0 & M_{13} = 0 \\ M_{21} \in V & M_{22} \in 1 \otimes_{k(x)} k(x^{1/p}) & M_{23} = 0 \\ M_{31} \in R & M_{32} \in 1 \otimes_{k(x)} k(x^{1/p}) & M_{33} \in 1 \otimes_{k(x)} k(x^{1/p}) . \end{array}$$

It is obvious that Σ is an Artinian ring and its radical N' consists of all matrices M in Σ for which $M_{11} = M_{22} = M_{33} = 0$.

Let Λ be Σ/N'^2, then one easily verifies that:

(a) $\text{gl.}\dim \Sigma = 1$

(b) $\text{gl.}\dim \Lambda = 2$

(c) Λ admits a splitting

(d) Σ does not admit a splitting (since V is not an R-direct summand in R).

From (b) and (c) it follows that $\Omega = \Gamma + A + A \otimes_\Gamma A$—with $A = N'/N'^2$—is a semi-primary hereditary ring ($A \otimes_\Gamma A \otimes_\Gamma A = 0$) with radical $M = A + A \otimes_\Gamma A$. Also $\Lambda = \Gamma + A$ and $A^2 = 0$. Therefore $\text{gl.}\dim \Omega = \text{gl.}\dim \Sigma = 1$, $\Omega/M^2 \approx \Sigma/N'^2 \approx \Lambda/N^2$ (N is the radical of Λ).