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SUPERATOMIC BOOLEAN ALGEBRAS

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A Boolean algebra B can be imbedded in a complete Boolean algebra C in such a way that all homomorphisms of B into complete Boolean algebras can be extended to complete homomorphisms on C if and only if B is superatomic, that is, every homomorphic image of B is atomic. This paper is a study of the structure of superatomic Boolean algebras, through the development of techniques of construction and classification, and through the representation of these algebras as compact Hausdorff clairsemé spaces.

It is shown that the weak direct product of any family of superatomic Boolean algebras is superatomic, and that any Boolean algebra generated by the union of a finite family of superatomic subalgebras is superatomic. The number of nonisomorphic superatomic Boolean algebras of infinite cardinality $\aleph$ is shown to be greater than $\aleph$. A complete algebraic and topological description of the countable superatomic Boolean algebras is given.

The concept of a superatomic Boolean algebra, that is, a Boolean algebra each of whose homomorphs is atomic, was first studied by Mostowski and Tarski in their work on Boolean algebras with ordered bases, [5]. Their results on these algebras are summarized in conditions (a)-(b′) of Theorem 1. The superatomic Boolean algebras again arose in the study of free extensions of Boolean algebras. It was proven by F. M. Yaqub ([8]) that, if $\alpha \geqq 2^{\aleph_0}$, then the free α-extension of a Boolean algebra B is α-representable if and only if B is superatomic. In considering the question of the existence of free complete extensions of Boolean algebras, it was found ([1]) that a Boolean algebra B has a free complete extension if and only if B is superatomic. Other results are summarized in the following theorem:

THEOREM 1. *If B is a Boolean algebra, then the following conditions are equivalent*:

(a) *B is superatomic; i.e., every homomorph of B is atomic.*

(a′) *No homomorph of B is atomless.*

(b) *Every subalgebra of B is atomic.*

(b′) *No subalgebra of B is atomless.*

(c) *No subalgebra of B is an infinite free Boolean algebra.*

(d) *B has no chain of elements order-isomorphic to the chain of rational numbers.*

(e) *$\mathscr{S}(B)$, the Stone space of B, is clairsemé; that is, every nonempty subspace of $\mathscr{S}(B)$ has at least one isolated point.*