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ON A THEOREM OF NIKODYM WITH APPLICATIONS TO WEAK CONVERGENCE AND VON NEUMANN ALGEBRAS

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The theorem of the title is a "striking improvement of the principle of uniform boundedness" in the space of countably additive measures on a sigma algebra. It says that if a set T of countably additive measures μ on a sigma algebra S is pointwise bounded: $\sup_{\mu \in T} |\mu(E)| < \infty, E \in S$, then it is uniformly bounded: $\sup_{\mu \in T}(\sup_{E \in S} |\mu(E)|) < \infty$.

Notice that the content of Nikodym's theorem is not changed by assuming that T is a countable set and recall that a countably additive complex valued measure on a sigma algebra is bounded and finitely additive.

An elementary example is given which illustrates that the theorem can not be extended to the case of bounded and countably additive measures on an algebra of sets. Next the theorem is extended, via a "sliding hump" argument, to the case where the measures are bounded and finitely additive on a sigma algebra. Then, after some remarks concerning weak convergence, the extended theorem is applied to extend recent results of Aarnes for normal functionals on a von Neumann algebra to the general case.

In order to set the notation, let us begin by stating our extension of Nikodym's theorem [4, Th. 8, p. 309-311].

THEOREM. *If $\{\mu_n\}$ is a sequence of bounded and finitely additive measures on a sigma algebra S of subsets of a set X such that for each element E of S $\sup_n |\mu_n(E)| < \infty$, then $\sup_n(\sup_{E \in S} |\mu_n(E)|) < \infty$.*

As mentioned above, before establishing the theorem let us consider the following example. Suppose X is the set of nonnegative integers and a subset E of X is in S if either E or X-E is a finite subset of the positive integers. Let $\{\mu_n\}$ be defined on S by $\mu_n(E) = n$ if E is a finite set containing n, $u_n(E) = 0$ if E is a finite set not containing n, and $\mu_n(E) = -\mu_n(X - E)$. Then $\{\mu_n\}$ is a sequence of bounded and countably additive measures on S satisfying (i) for each $E \in S$ $\lim_n |\mu_n(E)| = 0$, but also (ii)

$$\lim_n (\sup_{E \in S} \mu_n(E)) = \infty .$$

Proof of theorem. Suppose on the contrary that

$$\sup_n (\sup_{E \in S} |\mu_n(E)|) = \infty .$$