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ALGEBRAS OF GLOBAL DIMENSION ONE WITH A FINITE IDEAL LATTICE

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Let A denote a finite-dimensional (associative) algebra over an algebraically closed field K. It is well known that A has global dimension zero if and only if A is the direct sum of a finite number of full matrix algebras over K. In this paper a specific representation is given for those algebras A which have global dimension one (or less) and have only a finite number of (two-sided) ideals. It is shown that every such algebra is isomorphic to a (contracted) semigroup algebra $K[S]$ over a subsemigroup S of the semigroup of all $n \times n$ matrix units $\{e_{ij}\} \cup \{0\}$ which (i) contains $e_{11}, \cdots, e_{nn}$ and (ii) contains e_{ij} or e_{ji} whenever there are h and k such that e_{hi}, e_{ik} and e_{hj}, e_{jk} are in S. Conversely, if S satisfies (i) and (ii) then $K[S]$ has global dimension one or less and has a finite ideal lattice.

We use the definitions and notation of Cartan-Eilenberg ([2], VI, 2) and Jans ([11], 4). If A is a finite-dimensional algebra then A is Noetherian and therefore l. gl. dim. $A =$ r. gl. dim. A. In this case one writes gl. dim. A for this number. It is perhaps worthwhile to point out that if A is over an algebraically closed field, then gl. dim. A is precisely dim. A, the so-called Hochschild dimension of A (see [2], p. 176) and [8]). In [10] Hochschild proved that dim. $A \leqq 1$ if and only if A is segregated in every extension, i.e., every exact sequence of (finite-dimensional) algebras $B \rightarrow A \rightarrow 0$ splits. In [12] Jans gives a structure theorem for this class of algebras. By the above comments, for algebraically closed fields Jans' theorem is in fact a structure theorem for algebras of global dimension one or less. Unfortunately, however, we are unable at this time to relate the results of this paper to those of Jans.

Harada [9] has also given a characterization of semiprimary rings of global dimension $\leqq 1$ which is in spirit somewhat related to the methods of this paper. But again we are unable to deduce our results from Harada's.

On the other hand, Barry Mitchell has pointed out to the author that part of the main theorem of this paper is an immediate corollary of his work on the global dimension of abelian categories, see [15], pp. 229 ff. Specifically, one infers immediately from Mitchell's results that if S is a subsemigroup of $\{e_{ij}\} \cup \{0\}$ which contains all e_{ii}, then gl. dim. $K[S] \leqq 1$ if and only if whenever e_{hi}, e_{ik} and e_{hj}, e_{jk} are in S then either e_{ij} or e_{ji} is also in S. In this paper, however, we prefer