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GENERALIZED FRATTINI SUBGROUPS OF FINITE GROUPS

J. C. BEIDLEMAN AND T. K. SEO

The purpose of this paper is to generalize some of the fundamental properties of the Frattini subgroup of a finite group. For this purpose we call a proper normal subgroup H of G a generalized Frattini subgroup if and only if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P, p is a prime, of L such that $G = HN_G(P)$. Here $N_G(P)$ is the normalizer of P in G. Among the generalized Frattini subgroups of a finite nonnilpotent group G are the center, the Frattini subgroup, and the intersection $L(G)$ of all self-normalizing maximal subgroups of G. The product of two generalized Frattini subgroups of a group G need not be a generalized Frattini subgroup, hence G may not have a unique maximal generalized Frattini subgroup.

Let H be a generalized Frattini subgroup of G and let K be normal in G. If K/H is nilpotent, then K is nilpotent. Similarly, if the hypercommutator of K is contained in H, then K is nilpotent. We consider the Fitting subgroup $F(G)$ of a nonnilpotent group G, and prove $F(G)$ is a generalized Frattini subgroup of G if and only if every solvable normal subgroup of G is nilpotent.

Now let H be a maximal generalized Frattini subgroup of a finite nonnilpotent group G. Following Bechtell we introduce the concept of an H-series for G and prove that if G possesses an H-series, then $H = L(G)$.

2. Notation The only groups considered here are finite.

If H is a subgroup of a group G, then H' is the commutator (derived) subgroup of H,

$H^{(k)} (k > 1)$ is the k-th commutator subgroup of H,

$H^x = x^{-1}Hx$ for each $x \in G$,

$Z(H)$ is the center of H,

$Z^*(H)$ is the hypercenter of H(i.e. the terminal member of the upper central series of H), $D(H)$ is the hypercommutator of H(i.e. the terminal member of the lower central series of H),

$\phi(H)$ is the Frattini subgroup of H,

$F(H)$ is the Fitting subgroup of H(i.e. the largest nilpotent normal subgroup of H),

$N_G(H)$ is the normalizer of H in G.

If H is a subset of a group G, then denote by $\langle H \rangle$ the subgroup of G generated by H.