PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 3, 1967

PRODUCTS OF POSITIVE DEFINITE MATRICES. I

C. S. BALLANTINE

For each positive integer n, this paper gives necessary and sufficient conditions (nasc) on a 2×2 real matrix S (of positive determinant) that S be a product of n positive definite real (symmetric 2×2) matrices. Also, when S is the product of (real 2×2) positive definite matrices $P_1, P_2, \cdots, P_n$, it is shown that $P_1, P_2, \cdots, P_n$, and S must satisfy a condition which roughly speaking measures by how much (depending on S) $P_1, P_2, \cdots, P_n$ must collectively differ from scalar matrices.

For $n = 1$ or 2, the abovementioned nasc are known. For $n = 3$, the nasc on

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is (besides that $ad - bc > 0$) that

$$(c - b)^2 > 4(ad - bc) \text{ whenever } a + d \leqq 0 .$$

Thus S is here, in a sense, bounded away from the set of negative definite matrices, but its characteristic values may be both negative, since $(a + d)$ and $(ad - bc)$ may be chosen arbitrarily (except of course that $ad - bc$ must be positive) and independently. For $n = 4$, the nasc on S is (besides that $ad - bc$ be positive) just that S not be a negative scalar matrix. For $n \geqq 5$, the nasc is just that $ad - bc$ be positive.

These results are extensions of [1, Th. 1] and are obtained, without much additional effort, from the mechanism used in [1, l. c].

2. Main results.

We follow the notation of [1]. Let

$$(1) \qquad T = \Delta^{-(1/2)} S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; ad - bc = 1 ,$$

where $\Delta = \det S$ (note the slight change in notation from the previous section). We call T the *unimodular part* of S. Let

$$(2) \qquad p = [(a + d)^2 + (c - b)^2]^{1/2} = [(a - d)^2 + (c + d)^2 + 4]^{1/2} ,$$

$$(3) \qquad \zeta = \tfrac{1}{2} p e^{i\beta} = \tfrac{1}{2}[(a + d) + i(c - b)] \qquad (|\beta| \leqq \pi) ,$$

$$(4) \qquad z = [(a + d) + i(c - b)]^{-1}[(a - d) + i(c + b)] ,$$

$$(5) \qquad \rho = 2 \text{ argtanh } |z| = 2 \text{ argcosh} \left(\frac{p}{2}\right)$$