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# INITIAL SEGMENTS OF DEGREES

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**Spector first constructed a function $h$ whose degree of recursive unsolvability is minimal—that is to say that any function recursive in $h$ is either recursive or of the same degree as $h$. Define a set $Q$ of degrees to be an initial segment of the upper semi-lattice of degrees of unsolvability if**

$$a \in Q \wedge b < a \;.\!\rightarrow\!.\; b \in Q \;.$$

**Spector's result can then be interpreted as saying that a certain partially ordered set occurs as an initial segment of the degrees; it was conjectured that the same is true for every finite partially ordered set which has a least member. Sacks then constructed two minimal degrees $a$ and $b$ such that $a \cup b$ has $a, b, 0$ as its only predecessors.**

**In this paper their methods are extended to obtain the following result. Let $T$ be the upper semi-lattice of all finite subsets of $N$. Then $T$ can be embedded as an initial segment of the degrees. From this it follows that any finite partial ordering which can be embedded as an initial segment of $P(B)$ (the power set of $B$), with $B$ finite, can also be embedded as an initial segment of the degrees.**

We will define a function $h$ containing a countable infinity of functions $h_i$, each of minimal degree, such that $\underline{h}_{i_1} \cup \underline{h}_{i_2} \cup \cdots \cup \underline{h}_{i_t}$ will represent the finite subset $\{i_1, \cdots, i_t\}$ of $N$.

We first define a recursive function $\phi$ as follows: Let $\psi(k) = (\mu n)((n+1)((n+1)+1)/2) > k$ so that

$$\frac{\psi(k)(\psi(k)+1)}{2} \leqq k < \frac{(\psi(k)+1)((\psi(k)+1)+1)}{2} \,,$$

and define $\phi(k) = k - (\psi(k)(\psi(k)+1))/2$. Then $\phi(k)$ takes on successively the values

$$0, 0, 1, 0, 1, 2, 0, 1, 2, 3, \cdots, n, 0, 1, 2, \cdots, n, n+1, 0, \cdots .$$

We will want to arrange things so that in the $k$'th interval of $g$, the $\phi(k)$'th function (carried on the powers of $p_{\phi(k)}$) will be the *only* one for which $f_0$ and $f_1$ have different values. (The reader who finds this sentence mysterious is encouraged to read the next few definitions and then return to this remark.)

Let $p_i$ be the $i$'th prime, and define recursive predicates $P_i$ and $P$ as follows:

$$P_i(x) \equiv x = p_i^{(x)_i}$$
$$P(x) \equiv (Ei)P_i(x) \,.$$