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LIOUVILLE'S THEOREM ON FUNCTIONS WITH ELEMENTARY INTEGRALS

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Defining a function of one variable to be elementary if it has an explicit representation in terms of a finite number of algebraic operations, logarithms, and exponentials, Liouville's theorem in its simplest case says that if an algebraic function has an elementary integral then the latter is itself an algebraic function plus a sum of constant multiples of logarithms of algebraic functions. Ostrowski has generalized Liouville's results to wider classes of meromorphic functions on regions of the complex plane and J. F. Ritt has given the classical account of the entire subject in his Integration in Finite Terms, Columbia University Press, 1948. In spite of the essentially algebraic nature of the problem, all proofs so far have been analytic. This paper gives a self contained purely algebraic exposition of the problem, making a few new points in addition to the resulting simplicity and generalization.

A *differential ring* is a commutative ring R together with a *derivation* of R into itself, that is, a map $R \to R$ which, if denoted $x \mapsto x'$, satisfies the two rules

$$(x + y)' = x' + y'$$
$$(xy)' = x'y + xy' .$$

A *differential field* is a differential ring that is a field.

If u, v are elements of a differential field and $v \neq 0$ we have the relation $(u/v)' = (u'v - uv')/v^2$. In a differential ring we have $(x^n)' = nx^{n-1}x'$ for $n = 1, 2, 3, \cdots$. In particular, setting $x = 1, n = 2$, we have $1' = 0$. Elements of derivative zero are called *constants*, and in a differential field the totality of constants is itself a field, the *subfield of constants*.

If u, v are elements of a differential field such that $v \neq 0$ and $u' = v'/v$, in analogy with the classical situation we say that u is *a logarithm of* v or that v is *an exponential of* u. If, in a certain differential field, v has a logarithm, then it is necessarily unique to within an additive constant, while if u has an exponential, the latter is necessarily unique to within multiplication by a nonzero constant. A differential extension field of a differential field is said to be *elementary* if there exists a finite tower of intermediate differential fields, starting with the given small field and ending with the given extension field, such that each field in the tower after the first is