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SEQUENCES OF HOMEOMORPHISMS WHICH CONVERGE TO HOMEOMORPHISMS

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A technique often used in topology involves the inductive modification of a given mapping in order to achieve a limit mapping having certain prescribed properties. The following definition will facilitate the discussion. Suppose X and Y are topological spaces, and $\{W_i\}, i = 1, 2, \cdots,$ is a countable collection of subsets of X. Then a sequence $\{f_i\}, i \geqq 0,$ of mappings from X into Y is called stable relative to $\{W_i\}$ if $f_i \mid (X - W_i) = f_{i-1} \mid (X - W_i), i, = 1, 2, \cdots$. Note, in the above definition, that if $\{W_i\}$ is a locally finite collection, then $\lim_{i \to \infty} f_i$ is necessarily a well defined mapping from X into Y, and is continuous if each f_i is continuous. In a typical smoothing theorem, a C^r-mapping $f: M \to N$ between C^∞ differentiable manifolds M and N is approximated by a C^∞-mapping $g: M \to N$, where the mapping g is constructed as the limit of a suitable sequence $\{f_i\}$(with $f_0 = f$) which is stable relative to a locally finite collection $\{C_i\}$ of compact subsets of M. On the other hand, instead of improving f, it is also of interest to approximate f by a mapping g which has bad behavior at, say, a dense set of points of M. In this paper, such a mapping g is constructed as the limit of a sequence $\{f_i\}$ (with $f_0 = f$) which is stable relative to $\{C_i\}$, but where the C_i are more "clustered" than a locally finite collection. The case of interest here is where a sequence of homeomorphisms $\{H_i\}$, which is stable relative to $\{U_i\}$, necessarily converges to a homeomorphism. Theorem 1 of this paper gives a sufficient condition that the latter be satisfied for homeomorphisms of a metric space. In Theorem 1, the collection $\{U_i\}$ is not, in general, locally finite (in fact, the U_i satisfy a certain "nested" condition). Theorem 1 is used to establish a result concerning the distribution of homeomorphisms (of a differentiable manifold) which have a dense set of spiral points.

Let M be a metric space with metric d. We denote the (open) ball, of radius r, and centered at the point $x \in M$, by $B(x, r) = \{y \in M \mid d(x, y) < r\}$. The diameter of a nonempty subset A of M is $\delta(A) = \sup \{d(x, y) \mid x \in A, y \in A\}$. When M is euclidean n-space E^n, we write the points of E^n as $x = (x^1, \cdots, x^n)$, and provide E^n with the usual euclidean norm and metric

$$\| x \| = \left[\sum_{i=1}^{n} (x^i)^2 \right]^{1/2}, \quad d(x, y) = \| x - y \| .$$

The boundary of $B(x, r)$ in E^n is the $(n - 1)$-sphere $S(x, r) =$