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PRODUCTS OF POSITIVE DEFINITE MATRICES. II

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This paper is concerned with the problem of determining, for given positive integers n and j, which $n \times n$ matrices (of positive determinant) can be written as a product of j positive definite matrices. In § 2 the 2×2 complex case is completely solved. In particular, it turns out that every 2×2 complex matrix of positive determinant can be factored into a product of five positive definite Hermitian matrices and, unless it is a negative scalar matrix, can even be written as a product of four positive definite matrices. Sections 3 and 4 deal with the general $n \times n$ case. In § 3 it is shown that a scalar matrix λI can be written as a product of four positive definite Hermitian matrices only if the scalar λ is real and positive, and that λH (λ complex, H Hermitian) can be written as a product of three positive definite matrices only if λH is itself positive definite. In § 4 it is shown that every $n \times n$ real matrix of positive determinant can be written as a product of six positive definite real symmetric matrices and that every $n \times n$ complex matrix of positive determinant can be written as a product of eleven positive definite Hermitian matrices.

The 2×2 real case was earlier solved in [1, Th. 1 and the remarks immediately following]. The results in § 4 use only the 2×2 results and certain well known results. In later papers of this series the results of § 4 will be improved upon, using more refined methods.

In the rest of this section we state without proof several well known results that we shall use in later sections. First we introduce some notation. For a fixed positive integer n we denote by $\mathscr{H}$ the set of all $n \times n$ Hermitian matrices and by $\mathscr{P}$ the set of all positive definite matrices in $\mathscr{H}$. Then for each positive integer j we denote by $\mathscr{P}^j$ the set consisting of every $n \times n$ complex matrix which can be written as a product of j matrices from $\mathscr{P}$. (Thus $\mathscr{P}^1 = \mathscr{P}$.) Analogously, we denote by $\mathscr{K}$ the set of all $n \times n$ real symmetric matrices (thus $\mathscr{K}$ is just the set of all real matrices of $\mathscr{H}$), by $\mathscr{R}$ the set of all positive definite matrices of $\mathscr{K}$ (thus $\mathscr{R}$ is just the set of all real matrices of $\mathscr{P}$), and by $\mathscr{R}^j$ the set consisting of every $n \times n$ real matrix which can be written as a product of j matrices from $\mathscr{R}$ (so $\mathscr{R}^1 = \mathscr{R}$). For a matrix S all of whose eigenvalues are real the *inertia* of S is the ordered triple (of non-negative integers) consisting of the number of positive eigenvalues