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CONVOLUTION OPERATORS ON $L^p(G)$ AND PROPERTIES OF LOCALLY COMPACT GROUPS

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A locally compact group G is said to have property (R) if every continuous positive-definite function on G can be approximated uniformly on compact sets by functions of the form $s*\tilde{s}$, $s \in \mathscr{K}(G)$. When μ is a bounded, regular, Borel measure on G, the convolution operator T_μ defined by

$$(T_\mu)(s) = (\mu * s)(x) = \int_G s(y^{-1}x)d\mu(y) , \qquad s \in \mathscr{K}(G) ,$$

can be extended to a bounded operator on $L^p(G)$ whose norm satisfies $\| T_\mu \|_p \leqq \| \mu \|$. In this paper three characterizations of property (R) are given in terms of the norm $\| T_\mu \|_p$, $1 < p < \infty$, for specific operators T_μ. From these characterizations some closely-related, but seemingly weaker properties than (R), are shown to be equivalent to (R). Examples illustrating the results are given also.

If dx denotes left-invariant Haar measure on G and $\mathscr{K}(G)$ the space of continuous, complex-valued functions with compact support on G, the Haar modulus Δ is defined by

$$\int_G s(xa^{-1})dx = \Delta(a)\int_G s(x)dx , \qquad s \in \mathscr{K}(G) .$$

The Haar measure of a set $A \subset G$ is written $m(A)$. The norms on the measure algebra $M(G)$ and on the spaces $L^p(G)$, $1 \leqq p \leqq \infty$, defined with respect to the given Haar measure, will be denoted by $\|(\,.\,)\|$, $\|(\,.\,)\|_p$ respectively. For any space $\mathscr{D}(G)$ of functions or measures on G, the nonnegative elements in $\mathscr{D}(G)$ will be specified by $\mathscr{D}^+(G)$. We set $\tilde{s}(x) = \overline{s(x^{-1})}$, $s(x) = \overline{s(x^{-1})}\Delta(x^{-1})$ when $s \in \mathscr{K}(G)$ and $\mu^*(x) = \overline{\mu(x^{-1})}$ when $\mu \in M(G)$. Since $\mu \to \mu^*$ is an involution on $M(G)$, a measure μ is called hermitian if $\mu = \mu^*$. Following Godement ([8], see also Dixmier [5] § 13) we say that a measure $\mu \in M(G)$ is of positive type if

$$(1) \qquad \mu(s * \tilde{s}) = \int_G \left(\int_G \overline{s(x^{-1}y)}s(y)dy \right) d\mu(x) \geqq 0 ,$$

for all $s \in \mathscr{K}(G)$. When $(\,.\,,\,.\,)$ denotes the usual inner product on $L^2(G)$, inequality (1) can be rewritten as

$$(\mu * s, s) \geqq 0 , \qquad s \in \mathscr{K}(G) ,$$

changing s to $\tilde{s}$, i.e., μ is a positive element in the operator algebra