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BISECTION INTO SMALL ANNULI

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In a Riemannian manifold the modulus of a relatively compact set with border consisting of two sets of components is introduced to measure its magnitude from the viewpoint of harmonic functions. The existence of a subdivision into two sets each having modulus arbitrarily close to one is established.

1. Let M be a Riemannian manifold, i.e. a connected orientable C^∞ n-manifold that carries a metric tensor g_{ij}. Consider a bordered compact region $E \subset M$ whose border is the union of two nonempty disjoint sets α and β of components. We shall call the configuration (E, α, β) an *annulus*.

Let h be the harmonic function on E with continuous boundary values 0 on α and $\log \mu > 0$ on β such that

$$\int_\alpha *dh = 2\pi \ . \tag{1}$$

The number $\mu > 1$ is called the *modulus* of the annulus (E, α, β) and we set

$$\mu = \mathrm{mod}(E, \alpha, \beta) \ .$$

Let w be the *harmonic measure* of β with respect to E, i.e. the harmonic function on E with continuous boundary values 0 on α and 1 on β. By using Green's formula we obtain

$$\log \mu = \frac{2\pi}{D_E(w)} \ , \tag{2}$$

where $D_E(w)$ denotes the Dirichlet integral $\int_E dw \wedge *dw$ of w over E.

An illustration of these concepts is obtained by taking the annulus $E = \{x \mid r \leqq |x| \leqq R\}$ in n-dimensional $(n \geqq 3)$ Euclidean space. The harmonic measure of $|x| = R$ with respect to E is

$$w = \frac{|x|^{2-n} - r^{2-n}}{R^{2-n} - r^{2-n}}$$

and the modulus of $(E, |x| = r, |x| = R)$ is given by

$$\log \mu = \pi^{1-(n/2)}(2-n)\Gamma\left(\frac{n}{2}\right)(R^{2-n} - r^{2-n}) \ .$$

Note that $\mu > 1$, in a sense, measures the relative thickness of E and that $\mu \to 1$ as $R - r \to 0$.