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A NOTE ON CERTAIN BIORTHOGONAL POLYNOMIALS

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Konhauser has introduced two polynomial sets $\{Y_n^c(x; k)\}$, $\{Z_n^c(x; k)\}$ that are biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$. An explicit expression was obtained for $Z_n^c(x; k)$ but not for $Y_n^c(x; k)$. An explicit polynomial expression for $Y_n^c(x; k)$ is given in the present paper.

1. Konhauser [2] has discussed two sets of polynomials $Y_n^c(x; k)$, $Z_n^c(x; k)$, $n = 0, 1, \cdots, k = 1, 2, 3, \cdots, c > -1$; $Y_n^c(x; k)$ is a polynomial in x while $Z_n^c(x; k)$ is a polynomial in x^k. Moreover

$$(1) \qquad \int_0^\infty e^{-x}x^c Y_n^c(x; k)x^{ki}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n) \end{cases}$$

and

$$(2) \qquad \int_0^\infty e^{-x}x^c Z_n^c(x; k)x^{i}dx = \begin{cases} 0 & (0 \leqq i < n) \\ \neq 0 & (i = n) . \end{cases}$$

For $k = 1$, conditions (1) and (2) reduce to the orthogonality conditions satisfied by the Laguerre polynomials $L_n^c(x)$.

It follows from (1) and (2) that

$$(3) \qquad \int_0^\infty e^{-x}x^c Y_i^c(x; k)Z_j^c(x; k)dx = \begin{cases} 0 & (i \neq j) \\ \neq 0 & (i = j) . \end{cases}$$

The polynomial sets $\{Y_n^c(x; k)\}, \{Z_n^c(x; k)\}$ are accordingly said to be biorthogonal with respect to the weight function $e^{-x}x^c$ over the interval $(0, \infty)$.

Konhauser showed that

$$(4) \qquad Z_n^c(x; k) = \frac{\Gamma(kn + c + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + c + 1)}$$

As for $Y_n^c(x; k)$, he showed that

$$(5) \qquad \begin{aligned} Y_n^c(x; k) &= \frac{k}{2i} \int_C \frac{e^{-xt}(t + 1)^{c+kn}}{[(t + 1)^k - 1]^{n+1}} dt \\ &= \frac{k}{n!} \frac{\partial^n}{\partial t^n} \left\{ \frac{e^{-xt}(t + 1)^{c+kn}t^{n+1}}{[(t + 1)^{k+1} - 1]^{n+1}} \right\}_{t=0} . \end{aligned}$$

In the integral in (5), C may be taken as a small circle about the origin in the t-plane.