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ON SYMMETRY IN CERTAIN GROUP ALGEBRAS

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A complex Banach algebra A with involution $x \to x^*$ is *symmetric* if Sp $(x^*x) \subset [0, \infty)$ for each $x \in A$. It is shown that (i) if A is symmetric, the algebra of all $n \times n$ matrices with elements from A is symmetric, and (ii) the group algebra of any semi-direct product of a finite group with a locally compact group having a symmetric group algebra is again symmetric.

An involution $x \to x^*$ in A is said to be *hermitian* if Sp $(x) \subset (-\infty, \infty)$ for every self-adjoint $x \in A$. In [1] R. Bonic studied the natural involution in the group algebra of certain discrete groups and raised the question: *Is the group algebra of a semi-direct product of a finite group with a discrete Abelian group necessarily symmetric?* The present work is devoted to proving the more general result that the group algebra of any semi-direct product of a finite group with a locally compact group whose group algebra is symmetric, is again symmetric. The proof in part depends upon showing that the algebra of $n \times n$ matrices with elements from a symmetric Banach algebra has a naturally defined symmetric involution. (We restrict our attention to continuous involutions.)

I am indebted to the referee for pointing out that if G is discrete, our Theorem 2 follows from a result of A. Hulanicki (Corollary 2, page 286 of [4]). Also, while it is easy to show that every symmetric involution is necessarily hermitian and that the notions are equivalent for commutative algebras, the equivalence for noncommutative algebras was an open question until quite recently. Mr. S. Shirali has announced a positive solution to this question which will be contained in his Doctoral Dissertation at Harvard University.

1. Algebras of matrices. Let A be a Banach algebra with a continuous involution $x \to x^*$. A linear functional f on A is *positive* if $f(x^*x) \geqq 0$ for all $x \in A$. If A contains an identity e, such a functional satisfies $f(y^*x) = f(x^*y)$ for all $x, y \in A$, and if A is symmetric, then

$$\mathrm{Sp}(x) \subset \{f(x) \mid f \text{ a positive functional, } f(e) = 1\} \tag{1.1}$$

whenever $x \in A$ and $x^*x = xx^*$. (For a proof of these and other facts about symmetric Banach algebras, see [5].) In the following, $\nu(x)$ denotes the spectral radius of x.