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NOTE ON AN EXTREME FORM

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The purpose of this paper is to find a positive definite quadratic form $f_n(x_1, x_2, \cdots x_n)$ which is extreme and for which each of the binary form $f_2(x_i, x_j)$ is an extreme form. In other words we intend to seek an extreme n-ary form $f_n(x_1, x_2, \cdots x_n)$ which remains extreme when it is reduced to a binary form $f_2(x_i, x_j)$, by setting all but two of the x's equal to zero.

Let $f_n(x_1, x_2, \cdots x_n)$ be a quadratic form in n variables,

$$x_1, x_2, x_3 \cdots x_n \quad : f_n(x_1, x_2, \cdots x_n) = \sum a_{ij}x_ix_j \tag{1.1}$$

with determinant $D = |a_{ij}|$ and $a_{ij} = a_{ji} \cdot f_n(x_1, x_2 \cdots x_n)$ is positive-definite that is the roots of the characteristic equation

$$|a_{ij} - \lambda\delta_{ij}| = 0 \tag{1.2}$$

are all positive, where

$$\delta_{ij} = 1 \quad \text{if} \quad i = j;\ \delta_{ij} = 0 \quad \text{if} \quad i \neq j\ .$$

Let M denote the minimum value of $f_n(x_1, x_2 \cdots x_n)$ for integers $x_1, x_2, \cdots x_n$, not all zero. This M is the same for all forms derived from $f_n(x_1, x_2 \cdots x_n)$ by unimodular linear transformations. Let $2s$ denote the number of times this minimum is attained that is the number of solutions of the Diophantine equation:

$$f_n(x_1, x_2, \cdots, x_n) = M \tag{1.3}$$

Let $2s$ sets of (1.3) be given by

$$X = \pm M_k = \pm(m_{1k}, m_{2k}, \cdots, m_{nk}) \tag{1.4}$$

(known as minimal vectors) where $k = 1, 2, \cdots, s$.

Taking one of the two sets, considered not distinct, we have

$$\begin{gathered} \sum a_{ij}m_{ik}m_{jk} = M \\ k = 1, 2, \cdots, s\ . \end{gathered} \tag{1.5}$$

We consider (1.5) as equations in a_{ij} and suppose that (1.5) has an infinitude of sets of solutions in a_{ij}. This means that the auxiliary equation

$$\sum p_{ij}x_ix_j = 0 \tag{1.6}$$

with $p_{ij} = p_{ji}$, has an infinitude of sets of solutions.