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CURVATURE IN HILBERT GEOMETRIES II

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The interior of a closed convex curve C in the Euclidean plane can be given a Hilbert metric, which is preserved by projective mappings. Let p, q be points interior to C and let u, v be the points of intersection of the line pq with C. The Hilbert distance $h(p, q)$ is defined by

$$h(p, q) = \left| \log \frac{d(u, p)d(v, q)}{d(v, p)d(u, q)} \right| ,$$

where $d(x, y)$ denotes Euclidean distance. If C contains at most one line segment then $h(p, q)$ is a proper metric and the metric lines are the open chords of C carried by the Euclidean lines. Following Busemann [1, p. 237], we define the (qualitative) curvature at a point p as positive or negative if there exists a neighborhood U of p such that for every $x, y \in U$ we have

$$2h(\bar{x}, \bar{y}) \geqq h(x, x) \quad \textbf{respectively} \quad 2h(\bar{x}, \bar{y}) \leqq h(x, y) ,$$

where $\bar{x}$ $\bar{y}$ are the Hilbert midpoints of p and x and of p and y respectively.

In an earlier paper [2] we proved that any point p at which the sign of the curvature is determined is a projective center of C; that is, there exists a projective transformation which maps p into an affine center of the image of C. We also stated the conjecture that a Hilbert geometry has no point of positive curvature. It is the purpose of this paper to prove that conjecture.

Let C be centrally symmetric about the origin O. We may further assume that C has vertical lines of support at its points of intersection with the x-axis. Thus we may describe the upper arc of C by $y = y(x)$ and the lower arc by $y = -y(-x)$. Consider the points

$$a = (\varepsilon x, \varepsilon y(x)) , \qquad b = (\varepsilon x, -\varepsilon y(-x)) , \qquad 0 < \varepsilon < 1 .$$

Then

$$(1) \qquad \begin{aligned} h(O, a) &= \log \frac{\sqrt{x^2 + y(x)^2} \cdot (1 + \varepsilon)\sqrt{x^2 + y(x)^2}}{(1 - \varepsilon)\sqrt{x^2 + y(x)^2} \cdot \sqrt{x^2 + y(x)^2}} \\ &= \log \frac{1 + \varepsilon}{1 - \varepsilon} = 2\varepsilon + \frac{2}{3} \varepsilon^3 + O(\varepsilon^5) . \end{aligned}$$

If $\bar{a} = (\lambda x, \lambda y(x))$ is the Hilbert midpoint of O and a, then according to (1) we have