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ON THE STRUCTURE OF PRINCIPAL IDEAL RINGS

THOMAS W. HUNGERFORD

In this paper all rings are supposed commutative with identity (and all ring direct sums are finite). We distinguish between a principal ideal domain (PID) and a principal ideal ring (PIR) which may not be an integral domain. Our chief purpose is to prove:

THEOREM 1. **Every principal ideal ring R is a direct sum of rings, each of which is the homomorphic image of a principal ideal domain.**

A number of special cases of this theorem are well known. In order to prove the general case one needs several facts about complete local rings and formal power series rings. For the reader's convenience, this background material is summarized in § 1; the theorem is proved in § 2. In the final section is shown a counterexample indicating that in a sense Theorem 1 is best possible.

1. Preliminaries.

DEFINITION 2. A Noetherian ring R which has a unique maximal ideal M is a *local ring*.

The following proposition, a proof of which is in Lambek [5, p. 34], is extremely useful for characterizing local rings.

PROPOSITION 3. The following conditions on a ring R are equivalent:
(1) R has a unique maximal ideal M;
(2) all nonunits of R are contained in a proper ideal M;
(3) the nonunits of R form an ideal M.
If R is a local ring, then $\bigcap_{i \geq 1} M^i = 0$; this fact is due to Krull [3, Th. 2] and is also proved in Nagata [6, p. 12]. Consequently we obtain

PROPOSITION 4. Let R be a local ring whose maximal ideal M is principal, $M = (a)$, then R is a principal ideal ring and every nonzero element of R can be written in the form $a^k u$, with u a unit.

Proof. Let $r \in R$. Since $\bigcap M^i = \bigcap (a^i) = 0$, there is a highest power of a dividing r, say $r = a^k u$. Then u must be a unit, since all nonunits are in M and $u \in M$ implies $a \mid u$, hence $a^{k+1} \mid r$. If J is any ideal in R, all elements of J can be written in the form $a^i u$, u a unit. Choose such an element with i minimal. It is then easy to