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A MAXIMUM PRINCIPLE AND GEOMETRIC PROPERTIES OF LEVEL SETS

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There are many results in function theory which relate the behavior of a function in the interior of a domain to its behavior on the boundary. A well known result of this sort is the theorem of study: if the map of the unit disc under a univalent analytic function $f(z)$ is convex, then the map of every concentric disc contained therein is also convex. This theorem has been generalized in many different directions including more general properties of univalent functions, and the convex and star-shaped properties for level surfaces of harmonic functions in E^3. The results for univalent functions depend basically upon Schwarz's lemma, while the results for level surfaces of harmonic functions have been shown previously by means of rather complicated forms of the maximum principle.

In § 1, we give a simple and direct proof of a very general theorem, depending upon a form of the maximum principle, which is then shown in § 2 to easily give the known results as well as several new ones. Some related new problems are discussed in § 3.

1. Main result.

THEOREM 1. *Let C_{0j} and $C_{1j}, j = 0, 1, \cdots, n$, be closed subsets of Euclidean m-space, E^m, such that $C_{0j} \cap C_{1j} = \emptyset$, and define*:

$$p = (P_1, \cdots, P_n), P_j \in E^m ,$$
$$A_j = \partial C_{0j}, B_j = \partial C_{1j}, D_j = E^m - (C_{0j} \cup C_{1j}) ,$$
$$C_0 = C_{01} \times \cdots \times C_{0n} \qquad C_1 = C_{11} \times \cdots \times C_{1n} .$$

Suppose continuous functions $f_j(P_j)$: $E^m \to E^1$ and $T(p)$: $E^{mn} \to E^m$ are given such that the following conditions are satisfied for $j = 1, \cdots, n$:

(i) *The sets C_{0j} and C_{1j} are level sets of $f_j(P_j)$ such that*

$$f_j(P_j) = \begin{cases} 0 \ \ for \ \ P_j \in C_{0j} \\ K \ \ for \ \ P_j \in C_{1j} \end{cases} \qquad \begin{matrix} j = 0, \cdots, n \\ K \in (0, \infty] . \end{matrix}$$

$0 < f_j(P_j) < K$ *for all* $P_j \in D_j, j = 0, 1, \cdots, n$

(ii) $H_j(p) \equiv f_j(P_j) - f_0(T(p))$ *takes its maximum over all*

$$p \in N_j \equiv \{p: P_j \in D_j,\ T(p) \in D_0, f_j(P_j) \leqq f_i(P_i), i \neq j\}$$

in the set $\partial M_j \cap \partial N_j$, where $M_j \equiv \{p: P_j \in D_j,\ T(p) \in D_0\}$.