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A GENERALIZED COROLLARY OF THE BROWDER-KIRK FIXED POINT THEOREM

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This paper generalizes a corollary, due to W. A. Kirk, of the F. E. Browder-W. A. Kirk fixed point theorem for nonexpansive self-mappings of closed, bounded, convex sets in uniformly convex Banach spaces.

F. E. Browder [1] and W. A. Kirk [4] have independently proved that if F is a closed, bounded, convex subset of a uniformly convex Banach space, and if T is a nonexpansive mapping from F into F, then T has a fixed point in F. The following corollary was proved by Kirk [4] and also by Browder and Petryshyn [2]: If E is a uniformly convex Banach space, and $T: E \to E$ is a nonexpansive mapping, and if for some $x_1 \in E$ the sequence $\{T^n x_1\}$ of Picard iterates of T is bounded, then T has a fixed point in E. Browder and Petryshyn also observed that if the nonexpansive mapping T has a fixed point in E, then for any $x_1 \in E$ the sequence $\{T^n x_1\}$ will be bounded. Outlaw and Groetsch [6] have recently announced the following extension of this corollary: If E is a uniformly convex Banach space, and $T: E \to E$ is a nonexpansive mapping, and $S_\lambda = \lambda I + (1-\lambda)T$ for a given $\lambda, 0 < \lambda < 1$, then T has a fixed point in E if and only if the sequence $\{S_\lambda^n x_1\}$ of Picard iterates of S_λ is bounded for each $x_1 \in E$. The purpose of this note is to show that this corollary and its extension are both special cases of a considerably more general corollary of the Browder-Kirk theorem.

W. R. Mann [5] introduced the following general iterative process: Suppose $A = [a_{np}]$ is an infinite real matrix satisfying (1) $a_{np} \geqq 0$ for all n, p, and $a_{np} = 0$ for $p > n$; (2) $\sum_{p=1}^{n} a_{np} = 1$ for each n; (3) $\lim_n a_{np} = 0$ for each p. If F is a closed convex subset of a Banach space E, and $T: F \to F$ is a continuous mapping, and $x_1 \in F$, then the process $M(x_1, A, T)$ is defined by

$$v_n = \sum_{p=1}^{n} a_{np} x_p, \quad x_{n+1} = T v_n, \quad n = 1, 2, 3, \cdots .$$

Various choices of the matrix A yield many interesting iterative processes as special cases. With A the infinite identity matrix, one gets the Picard iterates of T: $v_{n+1} = x_{n+1} = T v_n$, whence $v_{n+1} = T^n v_1 = T^n x_1$. With $0 < \lambda < 1$ and $A = [a_{np}]$ defined by $a_{np} = \lambda^{n-1}$ if $p = 1$, $a_{np} = \lambda^{n-p}(1-\lambda)$ if $1 < p \leqq n$, $a_{np} = 0$ if $p > n$, $n = 1, 2, 3, \cdots$, one gets $v_{n+1} = \lambda v_n + (1-\lambda) T v_n = S_\lambda v_n$, whence $v_{n+1} = S_\lambda^n v_1 = S_\lambda^n x_1$. If T is linear then an appropriate choice of A yields