PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 3, 1968

BASES IN HILBERT SPACE

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A sequence (x_i) of elements of a Hilbert space, $\mathscr{H}$, is a *basis* for $\mathscr{H}$ if every $h \in \mathscr{H}$ has a unique, norm-convergent expansion of the form $h = \sum a_i x_i$, where (a_i) is a sequence of scalars. The sequence is *minimal* if there exists a sequence $(y_i) \subset \mathscr{H}$ such that $(x_i, y_j) = \delta_{ij}$. Every basis is minimal, and the sequence (a_i) in the expansion of h (above) is given by $a_i = (h, y_i)$. In this paper, we restrict our attention to *real* Hilbert space.

We derive, from classical characterizations of bases in B-spaces, criterea for (x_i) to be a basis for $\mathscr{H}$, as well as for (x_i) to be minimal in $\mathscr{H}$. We show that the sequence is minimal if and only if there are sequences $(g_i) \subset \mathscr{H}$ whose Gram matrices have a prescribed form. Similar conditions are obtained for (x_i) to be a basis for $\mathscr{H}$.

Let (x_i) be a linearly independent sequence of elements of $\mathscr{H}$. Using the Gram-Schmidt process, one finds an orthonormal basis, (w_i), for the closed span, $[x_i]$ of the sequence (x_i). We assume throughout that $[x_i] = \mathscr{H}$. Then, we may write

$$x_i = \sum_{j=0}^{i} p_{ij} w_j ,$$

and

$$w_i = \sum_{j=0}^{i} q_{ij} x_j .$$

If we let P and Q denote the matrices (p_{ij}) and (q_{ij}), respectively, then each is lower triangular, and $PQ = QP = I = (\delta_{ij})$. It is a classical result that Q is the unique inverse of P.

For (x_i) to be minimal, we need a sequence (y_i) such that $(x_i, y_j) = \delta_{ij}$. It is easy to see that, formally, $y_i = \sum_{y=i}^{\infty} q_{ji} w_j$. Further, the sequence is minimal if and only if the distance from x_k to $[x_j], j \neq k$ is positive. Using these facts, we get the following theorem. The second part is similar to the characterization of minimality due to Foias and Singer [2].

THEOREM 1. *Let $H = (h_{ij})$ denote the Gram matrix of (x_i), i.e., $h_{ij} = (x_i, x_j)$. Then the sequence is minimal if and only if any of the following conditions holds:*

(a) *The matrix $R = Q^T Q$ exists.*

(b) *There exists a sequence, (δ_i), with $\delta_i > 0$ for all i, such*