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UNIVERSALLY WELL-CAPPED CONES

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A closed convex cone P is said to be universally well-capped if it contains a compact convex subset B such that $P\backslash B$ is convex and $P = \cup_{n=1}^{\infty} nB$. The compact convex sets which are universal caps of some cone are represented as the positive part of the unit ball of an ordered Banach dual space with the weak* topology. A characterization, involving the directedness of the unit ball, is given of those ordered Banach spaces whose dual cones are universally well-capped. An application is made to the Choquet boundary theory for subspaces of continuous functions on a compact Hausdorff space.

The notion of a cap of a convex set X was introduced by Choquet [2] for the case where X is a *cone* in a Hausdorff locally convex space E. Following Choquet, B is a *cap* of X if B is a compact convex subset of X for which $X\backslash B$ is convex. If each point of X is contained in a cap then X is said to be *well-capped*. An important property of well-capped closed convex sets is that they satisfy a Krein-Milman type theorem [1; 2].

If B is a cap of a cone Q such that $Q = \bigcup_{n=1}^{\infty} nB$ then B is called a *universal cap* of Q. It is shown in [1] that if X is closed, convex and well-capped then $X \times \{1\}$ generates a closed convex well-capped cone Q in the space $E \times R$. Each cap of X is associated with a cap of the cone Q and each cap B of Q is itself a universal cap of the extremal sub-cone $\bigcup_{n=1}^{\infty} nB$. The purpose of this paper is to give a characterization of those compact convex sets which are universal caps of some cone. As far as we know, this problem was first posed by Choquet in lectures given at the University of Washington in 1964.

A particular instance of a universally capped cone is the case where the cone has a compact base. It is shown by Klee [11] that this is equivalent to the cone being locally compact. The properties of locally compact cones have been studied in some detail. We refer specifically to the work of D. A. Edwards [5] in which he shows that a locally compact cone can be embedded in a Banach space with the weak* topology as the dual of a Banach space with an order-unit norm.

We note here that an analogous construction is possible for universally capped cones. We give a characterization of the sub-dual spaces that arise in this context in terms of an ordering property of the unit ball which we term approximate directedness (definition