PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 1, 1968

A CHARACTERIZATION OF ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS AND APPLICATIONS

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It is proved that a semi-local ring R is analytically unramified if and only if R is a subspace of a ring which is isomorphic to a finite direct sum of semi-local Dedekind domains. Applying this, it is proved that a local domain R is analytically irreducible if and only if R is a subspace of a local Dedekind domain, and this is true if and only if R is a subspace of every local domain which dominates R and which satisfies the altitude formula relative to R. A final application proves that an analytically unramified local domain is unmixed if and only if it satisfies the altitude formula.

All rings in this article are commutative rings with a unit, and the terminology is in general the same as that in [2]. In particular a ring R^* *dominates* a ring R in case R is a subring of R^*, each maximal ideal in R^* contracts to a maximal ideal in R, and each proper ideal in R is contained in a maximal ideal in R^*. A semi-local (Noetherian) ring R is a *subspace* of a semi-local ring R^* in case R^* dominates R and R is a subspace of R^* for the natural (Jacobson radical) topologies. A *Dedekind domain* is an integrally closed Noetherian domain of altitude one.

In § 2 it is proved that a semi-local ring R is analytically unramified if and only if R is a subspace of a ring which is isomorphic to a finite direct sum of semi-local Dedekind domains (Theorem 2.1). When R is analytically unramified, Theorem 2.2 associates with each ideal B contained in the Jacobson radical J of R a ring W such that (1) W is a quotient ring of a finitely generated ring over R, and (2) W is isomorphic to a finite direct sum of semi-local Dedekind domains. Further, if Rad $B = J$, then (3) R is a subspace of W, and (4) if N is a maximal ideal in W, then $\operatorname{trd}(W/N)/(R/N \cap R)$ is equal to the depth of one (uniquely determined by N) of the prime divisors of zero in the completion of R (Proposition 2.8). To prove Theorem 2.2 a number of preliminary lemmas are needed and among these results Lemmas 2.3 and 2.4 are of some interest in themselves although they follow quite readily from known results. Essentially the method of proof of Theorem 2.2 is a combination of the methods used by Rees in [5, 6, 7, 8] to prove the Valuation Theorem.

Applications of Theorem 2.2 are given in § 3, 4 and 5. Theorem 3.5 and Theorem 3.7 characterize the intersection of W and the total