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ON THE VARIATION OF THE BERNSTEIN POLYNOMIALS OF A FUNCTION OF UNBOUNDED VARIATION

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The behavior of the ordinary Bernstein polynomials, B_nf, for discontinuous functions f can be quite erratic. The purpose of this note is to give an example of a function f which is quite irregular on the rationals but such that the total variation, VB_nf of B_nf tends to zero with n.

It is known that if f is of bounded variation, then VB_nf tends to the variation of f taken over its points of continuity, [2 p. 25]. In [3] we consider arbitrary f, and give sufficient conditions for VB_nf to tend to zero in terms of the sums $\sum_{r=0}^{n} |f(r/n)|$. It is shown in [2 p. 28] that B_nf, for unbounded f, can behave unusually in terms of pointwise convergence to f. Here we construct a function, unbounded on the rationals in every subinterval of $[0, 1]$, and which has the property that B_nf converges in variation (and uniformly) to zero.

2. **Preliminaries.** The n-th Bernstein polynomial of the real function f on $[0, 1]$ is

$$B_nf \equiv \sum_{r=0}^{n} f\left(\frac{r}{n}\right)p_{nr}(x) , \tag{2.1}$$

where

$$p_{nr}(x) \equiv \binom{n}{r} x^r(1-x)^{n-r} , \qquad x \in [0, 1] .$$

Since B_nf depends only on rational values of f, we restrict ourselves to "skeletons," i.e., functions defined only on the rationals in $[0, 1]$, in the manner of [1]. We need the following facts:

(A) If $r = 1, \cdots, n-1$, then for all n,

$$P(n, r) \equiv \operatorname*{Max}_{[0,1]} p_{nr}(x) < Cn^{\frac{1}{2}}[r(n-r)]^{-\frac{1}{2}} \tag{2.2}$$

where C is an absolute constant [1].

(B) If a is a positive integer, then

$$P(an, ar) < 2a^{-\frac{1}{2}}P(n, r) \tag{2.3}$$

for each $n \geqq 2$ and $r = 1, \cdots, n-1$. ((A) and (B) are applications of Stirling's formula.)

(C) For all n and f