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CLOSED SYSTEMS OF FUNCTIONS AND PREDICATES

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In this paper we show that there is a one to one correspondence between systems of functions defined on a finite set A and systems of predicates defined on A. This result implies that a complete set of invariants for a universal algebra on A is given by predicates defined on A. Conversely functions on A provide a complete system of invariants for sets of predicates closed under conjunction, change of variable and application of the existential quantifier.

We begin in § 2 by giving a definition of closure for systems of functions and predicates. This is followed by a definition of commutivity of a function and a predicate which gives a correspondence between the two types of systems. In Theorems 1 and 2 of § 3 we show that the correspondence is a Galois connection. In Theorem 3 we consider sets of predicates closed under the existential quantifier and show that the corresponding systems are determined by functions defined for all values of the arguments. In Theorems 4 and 5 we include disjunction and then negation in the definition of closure of a set of predicates. We also require that equality be among the predicates. The corresponding systems consist of essentially first order functions and essentially first order permutations respectively. We conclude in § 4 with some comments on the infinite case and some general comments on these results.

2. Basic definitions. Associated with any subset of A^{n+1}, the set of all sequences of length $n+1$ with elements in A, is the n-th order function $f(x_1, \cdots, x_n)$ which may be many valued and may not be defined on all of A^n. A system of functions $\mathscr{L}$ is defined to be closed if the following conditions are satisfied:

(i) $\mathscr{L}$ is closed under composition.

(ii) If $f(x_1, \cdots, x_n) \in \mathscr{L}$ is associated with the subset $P \subset A^{n+1}$ then any $g(x_1, \cdots, x_n)$ associated with $Q \subset P$ is in $\mathscr{L}$.

(iii) For any n, $\mathscr{L}$ contains all functions f defined on A^n such that $f(x_1, \cdots, x_n) = x_i$.

In defining closed systems of predicates the author has the following model in mind. We are given a sequence $A_1, A_2, A_3, \cdots$ of sets of predicates, each A_i containing all subsets of A^i. For each A_i a set of operators isomorphic to $\mathscr{S}_i$ the symmetric group is given which maps A_i onto A_i. These correspond to permutations of the variables