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ON QF-1 ALGEBRAS

DENIS RAGAN FLOYD

Let A be a finite-dimensional associative algebra with identity over a field k, M an A-module which is finite-dimentional as a vector space over k, and $E = \mathrm{Hom}_k(M, M)$ the algebra of linear transformations on M. For $a \in A$. Let a_L denote the linear transformation of M given by $a_L(x) = ax$, for $x \in M$. Define the following subalgebras of E:

$$A_L = \{a_L: a \in A\}$$
$$C = \{f \in E: f(ax) = af(x) \text{ for each } a \in A, x \in M\}$$
$$D = \{f \in E: f(g(x)) = g(f(x)) \text{ for each } g \in C, x \in M\}\ .$$

Clearly, $A_L \subseteqq D$. Require M to be faithful. Then A is isomorphic to, and will be identified with, A_L. If $A = D$, it is said that the pair (A, M) has the double centralizer property.

A is called a QF-1 algebra if (A, M) has the double centralizer property for each faithful A-module M.

The following results in the theory of QF-1 algebras are obtained:

1. Let A be a commutative algebra over an arbitrary field. Then A is QF-1 if and only if A is Frobenius.

2. Let A be an algebra such that the simple left A-modules are one-dimensional. Suppose there exist distinct simple two-sided ideals A_1 and A_2 contained in the radical of A, and primitive idempotents e and f, such that $eA_kf \neq 0$, for $k = 1, 2$. Then A is not QF-1.

3. Let A be an algebra with the properties that the simple left A-modules are one-dimensional, and the two-sided ideal lattice of A is distributive. Then if A satisfies any one of the following conditions, it is not QF-1.

(a) There exist, for $r \geqq 2$, $2r$ distinct simple two-sided ideals A_{uv} contained in the radical, and primitive idempotents e_{i_u} and e_{j_v} for $1 \leqq u, v \leqq r$, satisfying $e_{i_u}A_{uv}E_{j_v} \neq 0$, where the index pair (u, v) ranges over the set

$$(1, 1),\ (2, 1),\ (2, 2),\ (3, 2),\ (3, 3),\ \cdots,\ (r, r-1),\ (r, r),\ (1, r)\ .$$

(b) There exist, for $r \geqq 1, 2r + 2$ distinct simple two-sided ideals A_{uv} and A^{ρ}_{v}, for $(u, v) = (1, 1), (1, 2), \cdots, (r-1, r-1), (r-1, r)$, and $(\rho, v) = (1, 1), (2, 1), (3, r)$, and $(4, r)$, and primitive idempotents $e_{,_u}, e_{j_v}$, and e_{k_ρ} satisfying $e_{i_u}A_{uv}e_j \neq 0$ and $e_{k_\rho}A^{\rho}_{v}e_{j_v} \neq 0$, where (u, v) and (ρ, v) range over the index pairs indicated above.

It is to be noted that the condition given in 2b is but one of three conditions of that type which may be formulated. An algebra