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ALGEBRAIC GEOGRAPHY: VARIETIES OF STRUCTURE CONSTANTS

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Let $\mathscr{C} = \mathscr{C}(n, \Omega)$ denote the algebraic set of structure constants for n-dimensional associative algebras, a subset of Ω^{n^3}. Here Ω is a universal domain over a prime field F and a point $c = (c_{hij})$ with $h, i, j = 1, \cdots, n$ is in $\mathscr{C}$ if and only if the multiplication $(x_h, x_i) \to x_h x_i = \sum_j c_{hij} x_j$ is associative. The set $\mathscr{C}$ is readily seen to be F-closed in the Zariski topology on Ω^{n^3} and is in fact a finite union of irreducible closed cones (the components of $\mathscr{C}$) with the origin as vertex. The natural "change of basis" action of the group $G = GL(n, \Omega)$ on $\mathscr{C}$ yields a one-one correspondence between orbits $G \cdot c$ on $\mathscr{C}$ and n-dimensional Ω-algebras. One studies the globality of these algebras (and of algebras defined over subfields of Ω) by examining the geography of $\mathscr{C}$.

Thus if S is a semi-simple Ω-algebra (more generally, if the Hochschild group $H^2(S, S) = (0)$) then its corresponding orbit (denoted $G \cdot S$) is open and therefore dense in its component $\mathscr{C}_0$ of $\mathscr{C}$. Thus S determines all algebras which live on $\mathscr{C}_0$. One checks that $\dim \mathscr{C}_0 = n^2 - n + s$, where $S = S_1 \oplus \cdots \oplus S_s$ for simple S_α. Moreover, in the language of Gerstenhaber and Nijenhuis-Richardson, one may hope to deform the algebras on $\mathscr{C}_0$ into S. In commencing a study of the parameter space $\mathscr{C}$, therefore it seems a natural first question to ask whether every irreducible component of $\mathscr{C}$ is dominated by such an open orbit or, in the sense of deformation theory, "Does every algebra deform into a rigid algebra?" We show here that the answer is no.

In § 1 below, we develop some relations between deformation of algebras and specialization of points on $\mathscr{C}$. In § 2 our question is partly settled by a demonstration that every component of $\mathscr{C}$ must carry an open subset of nonsingular points which is either the orbit of a single rigid algebra or an infinite union of orbits of Ω-algebras which differ only in their radicals. Then in § 3 we answer the question in the negative, showing that the second alternative of § 2 does in fact occur by exhibiting a full component of $\mathscr{C}$ which consists entirely of the orbits of three-dimensional nilpotent algebras.

The author recalls with pleasure several discussions with Professors Murray Gerstenhaber, Albert Nijenhuis, Alan Landman and Maxwell Rosenlicht during the course of this work.

1. Deformation and specialization.

Throughout this paper, k