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INTEGRAL INEQUALITIES INVOLVING SECOND ORDER DERIVATIVES

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An integral inequality involving second order derivatives is derived. A most important consequence of this inequality is that the Dirichlet form

$$D(u, u) = \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} = q|u|^2 dx \geqq 0 ,$$

for functions $q(x)$ which are positive and "not too large" in a sense which will be made precise later and for functions $u(x)$ with compact support contained in D. Some examples are given and an application is made to an existence theorem for a fourth order uniformly elliptic P.D.E.

An earlier paper by the author [1] contains some similar results for inequalities involving first derivatives. The following definitions and notations will be used throughout the paper. Let

$$x = (x_1, x_2, \cdots, x_n) \in R^n .$$

Let D be an open domain in R^n which may be unbounded. Let $C^\infty(D)$ denote the set of infinitely differentiable complex valued functions on D and let $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ consisting of functions with compact support contained in D. Let

$$\| u \|_q = \left(\int_D \sum_{i=1}^n | D_i^2 u |^2 + q| u |^2 dx\right)^{1/2}, \text{ where } D_i^2 u = \frac{\partial^2 u}{\partial x_i^2}$$

and q is either equal to 1 or to one of the positive functions to be defined later. Let $H_q(D)$ be the completion of $\{u \in C^\infty(D): \| u \|_q < \infty\}$ with respect to $\| u \|_q$ and let $\overset{\circ}{H}_q(D)$ be the completion of $C_0^\infty(D)$ with respect to $\| u \|_q$. The functions u in $H_q(D)$ or $\overset{\circ}{H}_q(D)$ have strong L_2 second derivatives which we will denote by the same symbol as for the oridnary derivative. So that

$$\lim_{n \to \infty} \int_D | D_i^2 u - D_i^2 u_n |^2 dx = 0$$

where $\{u_n\}$ is any sequence of elements in $C^\infty(D)$ such that $\| u - u_n \|_q \to 0$. All coefficient functions considered will be real valued. The variable functions u may be complex valued. There do not seem to be any analogues of the basic results with complex valued coefficients.

THEOREM 1. *Suppose that the boundary of D is smooth enough*