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ON THE JORDAN STRUCTURE OF COMPLEX BANACH *ALGEBRAS

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All algebras considered are complex Banach algebras with identity and continuous involution. The principal results of §1 are that for a Jordan *homomorphism T of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ where $\mathfrak{A}_2$ is *semisimple, continuity is automatic, the kernel is a closed *ideal, and if $\mathfrak{A}_2$ is commutative then the factor algebra $\mathfrak{A}_1$/kernel T is also commutative. In §2 a cone different from the usual cone is introduced and its relation to the usual cone is studied. The principal result is that if this cone coincides with the usual cone, then any Jordan *representation is the sum of a *representation and a *antirepresentation. §3 is devoted to proving that for a *semisimple algebra, the axiom $\|xy\| \leqq \|x\| \|y\|$ follows from the weaker axiom $\|xy + yx\| \leqq 2\|x\| \|y\|$.

Our notation will be as follows: $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc., are algebras; e, e_1, e' etc. are the identities of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc.; $\| \ \|_{sp}$ is the spectral radius; H, H_1, H' etc. are the real subspace of hermitian elements of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}'$ etc. We generally abbreviate "Jordan" as simply "J".

1. DEFINITION. A linear transformation $T: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is called a *J-homomorphism* if $T(xy + yx) = T(x)T(y) + T(y)T(x)$ and $T(e_1) = e_2$. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have involutions and $T(x^*) = T(x)^*$ then T is called a *J-*homomorphism*.

The assumption $T(e_1) = e_2$ will usually be used in studying the spectrum. Since the adjunction of an identity merely adjoins 0 to any spectrum which does not already include 0, this assumption is removable in many situations.

LEMMA 1.1 *If $x \in \mathfrak{A}_1$ then the spectrum of Tx is contained in the spectrum of x.*

Proof. It suffices to show that if x has a two-sided inverse in $\mathfrak{A}_1$ then Tx has a two-sided inverse in $\mathfrak{A}_2$. So let $yx = xy = e_1$, $u = Tx$ and $v = Ty$. Then $uv + vu = T(xy + yx) = T(2e_1) = 2e_2$. Multiply on each side by v separately to get $vuv + v^2u = 2v$, $uv^2 + vuv = 2v$ so that $v^2u = uv^2$. Therefore u commutes with v^2 and hence so does u^2. Thus $2u^2v^2 = 2v^2u^2 = u^2v^2 + v^2u^2 = T(x^2y^2 + y^2x^2) = T(2e_1) = 2e_2$. Therefore $u^2v^2 = v^2u^2 = e_2$ and hence $u = Tx$ has a two-sided inverse in $\mathfrak{A}_2$.

An immediate consequence of Lemma 1.1 is the following lemma