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SINGULAR INTEGRALS AND POSITIVE KERNELS

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Let $k(x, t)$ be a C^1 kernel of a positive integral operator on $L^2(E)$ where E is compact. It is shown that certain singular self-adjoint operators A on $L^2(E)$, consisting of the sum of a multiplication operator and a generalized Hilbert transform integral operator with Kernel $ik(x,t)(x-t)^{-1}$, are analogous to—sometimes even to the extent of unitary equivalence—operators, about which a good deal more is known, of the same structure as A but with $k(x, t)$ of the special form $\phi(x)\bar{\phi}(t)$.

Let E be a set of real numbers of positive measure and suppose that

(1.1) $h(x)$ is essentially bounded, real and measurable on E and that

(1.2) $k(x, t)$ is essentially bounded, measurable and satisfies $k(x, t) = \bar{k}(t, x)$ on $E \times E$. Then the transformation $f \rightarrow Af$, where

$$(1.3) \qquad (Af)(x) = h(x)f(x) + i\int_E k(x, t)(x-t)^{-1}f(t)dt ,$$

the integral being interpreted as a Cauchy principal value, is a bounded, self-adjoint operator on $L^2(E)$. Concerning such integrals, see Muskhelishvili [4], also Calderón [1], Putnam [7, 8], Schwartz [13, 14]. The spectral theory of A when $k(x, t)$ is of the form

$$(1.4) \qquad k(x, t) = \phi(x)\bar{\phi}(t) ,$$

where

(1.5) $\phi(x)$ is essentially bounded and measurable on E, has been extensively investigated. See, in particular, Koppelman [3], Pincus [5, 6], and Rosenblum [11], the latter containing a treatment of the theory for E arbitrary.

Let K denoted the self-adjoint integral operator defined on $L^2(E)$ by

$$(1.6) \qquad (Kf)(x) = \int_E k(x, t)f(t)dt, \qquad f \in D_K \subset L^2(E) .$$

An important property of K which will be studied in this paper is that of positivity, $K \geqq 0$, that is

$$(1.7) \qquad (Kf, f) \geqq 0 \qquad \text{for } f \in D_K .$$

It is seen that operators K with kernels $k(x, t)$ given by (1.4) satisfy