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FUNCTIONS REPRESENTED BY RADEMACHER SERIES

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A series of the form $\sum_{m=1}^{\infty} a_m r_m(t)$, where $\{a_m\}$ is a sequence of real numbers and $r_m(t)$ denotes the mth Rademacher function, sign $\sin(2^m \pi t)$, is called a Rademacher series (as usual, sign $0 = 0$).

Letting $f(t)$ denote the sum of this series whenever it exists, we shall investigate the effect that various conditions on $\{a_m\}$ have on the continuity, variation, and differentiability properties of f.

2. Continuity properties.

We now prove

THEOREM (2.1). *If $\sum |a_m| < \infty$, then $f(t)$ is continuous at dyadic irrationals* (i.e., *numbers not of the form $p/2^k$*) *and has right and left hand limits everywhere in* $[0, 1]$.

Proof. Under our hypothesis we have that $\sum a_m r_m(t)$ converges uniformly to $f(t)$, which implies our conclusion since the Rademacher functions are continuous at dyadic irrationals and have right and left hand limits everywhere in $[0, 1]$.

In general, the right and left hand limits of $f(t)$ are unequal at dyadic rationals. We now investigate under what conditions we have equality and prove.

THEOREM (2.2). *If $\sum |a_m| < \infty$, then the following are equivalent:*

(a) $a_k = \sum_{m=k+1}^{\infty} a_m$,

(b) $f(p2^{-k} + \varepsilon_n) \to f(p2^{-k})$ *as* $n \to \infty$,

(c) $f(p2^{-k} + \delta_n) \to f(p2^{-k})$ *as* $n \to \infty$,

(d) $f(p2^{-k} + \varepsilon_n) - f(p2^{-k} + \delta_n) \to 0$ *as* $n \to \infty$,

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are some positive and negative sequences tending to zero, and p is an odd integer.

Proof.

$$f(p2^{-k} + t) - f(p2^{-k}) = \sum_{m=1}^{k-1} a_m r_m(p2^{-k} + t) - a_k r_k(t) + \sum_{m=k+1}^{\infty} a_m r_m(t) - \sum_{m=1}^{k-1} a_m r_m(p2^{-k}) ,$$

since $r_m(p2^{-k} + t) = r_m(t)$ if $m \geqq k + 1$, and $r_k(p2^{-k} + t) = -r_k(t)$.