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A NOTE ON EBERLEIN'S THEOREM

CARL L. DEVITO

This paper is concerned with locally convex spaces which are closed, separable subspaces of their strong biduals. Let E be a space of this type. We first prove that, for an element of E'', weak* continuity on E' is equivalent to sequential weak* continuity on the convex, strongly bounded subsets of E'. We then prove Eberlein's theorem for spaces of this type; i.e., we prove that, for the weakly closed subsets of E, countable weak compactness coincides with weak compactness. Finally, we show that the separability hypothesis in our first theorem is necessary.

Our notation and terminology will be that of [1]. The letter E will always denote a locally convex, topological vector space over the field of real numbers. If we want to call attention to a specific, locally convex topology t on E, we will write $E[t]$. The dual of E will be denoted by E'. The weakest topology on E which renders each element of E' continuous will be denoted by $\sigma(E, E')$. We shall be working with the strong topology, $\beta(E', E)$, on E'. This is the topology of uniform convergence on the convex, $\sigma(E, E')$-bounded subsets of E. E'' will denote the dual of $E'[\beta(E', E)]$. We shall often identify E with its canonical image in E''. The topology induced on E by its strong bidual, $E''[\beta(E'', E')]$, will be denoted by $\beta^*(E, E')$. Recall that $\beta^*(E, E')$ is the topology of uniform convergence on the convex, $\beta(E', E)$-bounded subsets of E'.

DEFINITION. We shall say that E has property (S) if the following is true: An element w of E'' is in E if and only if $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of E' which is $\sigma(E', E)$-convergent to zero.

THEOREM 1. *Suppose that $E[\beta^*(E, E')]$ is separable. Then E has property (S) if and only if E is a closed, linear subspace of $E''[\beta(E'', E')]$.*

Proof. We shall prove sufficiency first. Let w be in E'' and suppose that $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of E' which is $\sigma(E', E)$-convergent to zero. Let B be a convex, $\beta(E', E)$-bounded subset of E' and let F be the dual of $E[\beta^*(E, E')]$. Clearly $E' \subset F$ and, by [1; Prop. 2, p. 65], B is relatively $\sigma(F, E)$-compact. Since E is $\beta^*(E, E')$-separable, the restriction of $\sigma(F, E)$ to B is metrizable. Hence $\sigma(E', E)$ is metrizable on every