PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

ON A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

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The purpose of this note is to comment on and extend recent results of C. Fefferman. A proof of his Radon-Nikodym theorem that is, perhaps, more amenable to generalization is given. A Lebesgue decomposition is also obtained.

Since the notations in [3] and [7] conflict, we shall make the following compromises in notation and terminology, and beg the reader's indulgence.

Let S be a set, Σ be an algebra of subsets of S, C be the complex numbers, and R be the real numbers. Let $H(C) = H(S, \Sigma: C)$ denote the set of all bounded, complex valued, finitely additive set functions on Σ. Then $H(R)$ will denote the real valued elements of $H(C)$. If $\alpha \in H(C)$ and $E \in \Sigma$, we denote the total variation of α over E by $v(\alpha, E)$. If $\alpha, \beta \in H(C)$ then

(i) α is absolutely continuous with respect to $\beta(\alpha \ll \beta)$ means: given $\varepsilon > 0$, there exists $\delta > 0$ such that $v(\beta, E) < \delta$ $(E \in \Sigma)$ implies $v(\alpha, E) < \varepsilon$.

(ii) α is singular with respect to $\beta(\alpha \perp \beta)$ means: given $\varepsilon > 0$, there exists $E \in \Sigma$ such that $v(\alpha, E) < \varepsilon$ and $v(\beta, S\text{-}E) < \varepsilon$.

The classical Radon-Nikodym theorem (eg., [6, Th. III. 10.2]) asserts that if Σ is a sigma algebra and λ is a countably additive element of $H(C)$, then λ can be given an integral representation with respect to a nonnegative, countably additive element μ of $H(R)$ if, and only if, λ is absolutely continuous with respect to μ.

In 1939, S. Bochner published a generalization ([1]) which removed the restrictions that Σ be a sigma algebra and that the set functions be countably additive. Then S. Bochner and R. S. Phillips [2] used a vector lattice approach to give a new proof of Bochner's Theorem and, also, to obtain a Lebesgue decomposition. S. Leader [8] studied the L^p-spaces associated with finitely additive measures. A representation for the case where $\mu \in H(R)$ appeared ([3]) in 1962. Theorem III. 10.7 of [6] supplements the classical theorem by allowing μ to be complex valued, and recently C. Fefferman ([7]) extended the latter result to the case of a general algebra of subsets of a set.

Let us turn to some comments on the paper of Fefferman.

(i) The definition of absolute continuity given in [7] seems to contain a misprint: Suppose that $S = [-1, 1]$, Σ is the sigma algebra of Lebesgue measurable subsets of S, α is Lebesgue measure m res-