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GROWTH TRANSFORMATIONS FOR FUNCTIONS ON MANIFOLDS

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In this paper we look at the problem of maximizing a function P defined on a manifold M. Although we shall be primarily concerned with the case where M is a certain polyhedron in a Euclidean space R^n and P is a polynomial with nonnegative coefficients defined on R^n, some of our results are valid in greater generality.

In § 2 we describe the general behavior of a growth transformation of P in the vicinity of a local extremum. These results are of a topological nature and can be thought of as a topological—dynamical description of growth transformations.

In § 3 we turn our attention to a particular class of growth transformation which arise for polynomials with nonnegative coefficients. We shall prove the following result, which is the main theorem of this paper:

THEOREM. *Let $M \cup \partial M$ denote the manifold with boundary given by $x = (x_{ij})$ where*

$$\left\{x_{ij} : x_{ij} \geqq 0 \text{ and } \sum_{j=1}^{q_i} x_{ij} = 1\right\}$$

where $q_1, \cdots, q_k$ is a set of nonnegative integers. Let P be a homogeneous polynomial in the variable $\{x_{ij}\}$, with nonnegative coefficients. Let $\mathscr{T} = \mathscr{T}_P : M \to M \cup \partial M$ defined by $y = \mathscr{T}_P(x)$ where

$$y_{ij} = x_{ij} \frac{\partial P}{\partial x_{ij}} \left[\sum_{k=1}^{q_i} x_{ik} \frac{\partial P}{\partial x_{ik}} \right]^{-1} .$$

Then

$$(1) \qquad P(x) \leqq P(t\mathscr{T}_P(x) + (1 - t)x) , \qquad (0 \leqq t \leqq 1, x \in M) .$$

The proof of this is based on a suitable modification of an argument of L. E. Baum and J. A. Eagon, cf., [1].

We also study the problem of extending the mapping $\mathscr{T}_P$ to the boundary ∂M in such a way that it is continuous. These results are stated in Theorem 7. It is a consequence of this that $\mathscr{T}_P$ maps neighborhoods of a local maximum into themselves even if the maximum is on the boundary.

In § 5 we examine other growth transformations that are related