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A UNIFYING CONDITION FOR IMPLICATIONS AMONG THE AXIOMS OF CHOICE FOR FINITE SETS

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For $n \geq 1$, let $C(n)$ be the axiom of choice restricted to sets of n-element sets. We define a condition, (Z), which is sufficient to assure the provability of an implication

$$(C(m_1) \,\&\, C(m_2) \,\&\, \cdots \,\&\, C(m_s)) \longrightarrow C(n)$$

in set theory. We compare condition (Z) with various other conditions related to the above implication.

1. **Notation and preliminaries.** Let σ be the set theory of [3]; this is a set theory of the Gödel-Bernays type which permits the existence of urelemente (objects, other than the null set, which are in the domain, but not the range, of the $\in$-relation) and which does include the axiom of choice among its axioms. Our independence statements will assume that σ is consistent; this is equivalent to the assumption that Gödel's system A, B, C, of [2], is consistent. Our logical framework is the first-order predicate calculus with identity.

By the nonnegative integers we mean the Von-Neumann integers, i.e., 0 is the empty set, $1 = \{0\}, 2 = 1 \cup \{1\}, 3 = 2 \cup \{2\}$, etc. For each such n, we let I_n be the set of all integers $\geqq n$ and we let J_n be the relative complement of I_{n+1} in I_1, $I_1 \backslash I_{n+1}$. We let Π represent the set of prime numbers, and we let $\Pi_n = \Pi \cap I_n$.

If there is a function (which is itself a set) which maps the set x one-one onto the positive integer n, then x is called an *n-element set*; in this case we let $n(x)$ denote the unique integer n for which such a mapping exists.

DEFINITION 1. For $n \in I_1$ let $C(n)$ denote the following statement of set theory: "For every set x of n-element sets there is a function f defined on x such that for each $y \in x, f(y) \in y$. The statements $C(n)$ are called the *axioms of choice for n-element sets* or simply the *axioms of choice for finite sets*.

For any set x let $\mathscr{P}(x)$ denote the power set of x and let $\mathscr{P}^{\#}(x)$ designate the set consisting of 0 together with the set of all n-element subsets of x for $n \in I_1$. For $Z \in \mathscr{P}^{\#}(I_1)$, let $C(Z)$ be the conjunction of the statements $C(z), z \in Z$. Since a positive integer is not a subset of I_1, no confusion will result from our usage of $C(n)$ instead of $C(\{n\})$.

We shall be concerned with implications of the form

$$(1) \qquad C(Z) \longrightarrow C(n)$$