PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

ON SUBGROUPS OF FIXED INDEX

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If $k \in \mathscr{K}$, where $\mathscr{K}$ is a subgroup of a group $\mathscr{S}$, then closure implies $k^2, k^3, \cdots, \in \mathscr{K}$. Nonempty subsets $S \subset \mathscr{S}$ with the inverse property $s^m \in S$ implies $s, s^2, \cdots, s^m \in S$ $(m = 1, 2, \cdots)$ will be called *stellar sets*. Let p^α be a fixed prime power. If a stellar set S of an *abelian* group $\mathscr{S}$ intersects every subgroup $\mathscr{K}$ of index p^α in $\mathscr{S}$, and $0 \notin S$, then the cardinal $|S|$ of S is bounded below by p^α (Theorem 3), when $\mathscr{S}$ satisfies a mild condition.

Hence for instance a subset S of euclidean n-space E_n intersecting all sublattices of determinant p^α of the fundamental lattice will have at least p^α elements, and more if no element is divisible by p^α.

Henceforth $\mathscr{S}$ will always be an additive abelian group, so a *stellar set* will be one with

$$\begin{gathered}\varnothing \neq S \subset \mathscr{S} \\ mg \in S \Rightarrow g, 2g, \cdots, mg \in S (g \in \mathscr{S}, m = 1, 2, \cdots)\,.\end{gathered} \tag{1}$$

Examples of stellar sets are $\mathscr{S}$ itself, and its *periodic part* [5, p. 137]; and a *star set* [7] is a symmetric stellar set. There are stellar sets of one element s, i.e., those s for which $s = mg (m = 1, 2, \cdots)$ implies $m = 1$. Now let p be a fixed prime, and suppose S intersects every subgroup $\mathscr{K}$ of $\mathscr{S}$ of index p. Suppose also

$$0 \notin S \tag{2}$$

(if $0 \in S$ the intersection property is redundant). Then we can say the following (in this paper we denote $|A| =$ cardinal of A, $mA = \{ma;\, a \in A\}$, for any set A and integer m):

THEOREM 1. *Let p be a fixed prime, $\mathscr{S}$ an abelian group, and S a stellar set with $0 \notin S$ which intersects all subgroups $\mathscr{K}$ of index $\mathscr{S} : \mathscr{K} = p$. Then*

$$|S| \geqq p\,. \tag{3}$$

When $S \cap p\mathscr{S} = \varnothing$ we have $|S| > p$.

A similar result holds for ordinary sets T:

THEOREM 2. *Suppose p is a fixed prime, $\mathscr{S}$ is an abelian group with more than one subgroup of index p, and T is any subset of $\mathscr{S}$ with*