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INVARIANT EXTENSIONS OF LINEAR FUNCTIONALS, WITH APPLICATIONS TO MEASURES AND STOCHASTIC PROCESSES

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A theorem is proved slightly stronger than the following. Let G be a set of order-preserving linear operators on a partially-ordered real linear space X, for which there exist sets $G = G_n \supseteqq G_{n-1} \supseteqq \cdots \supseteqq G_0$ with G_0 commutative and such that for $k = 1, \cdots, n$, x in X, g_1 and g_2 in G_k there exist h_1 and h_2 in G_{k-1} satisfying $h_1 g_1 g_2(x) = h_2 g_2 g_1(x)$. If S is a G-invariant subspace such that for all x in X there is an s in S satisfying $s \geqq x$, and f_0 is a G-invariant positive linear functional on S, then f_0 extends to a G-invariant positive linear functional on X. This is used to construct a generalized form of the Banach limit, an ergodic measure on compact Hausdorff spaces, a stationary extension of a relatively stationary stochastic process $x_t (0 \leqq t \leqq a)$ with values in an arbitrary space, and a generalization to arbitrary linear spaces of Krein's extension theorem for positive-definite complex-valued functions.

This paper consists chiefly of one principal theorem (Theorem 2 in § 1) on extending positive linear functionals from a subspace S of a linear space X to all of X so as to preserve invariance under a set G of order-preserving linear transformations, together with several applications of that theorem. The set G is assumed to satisfy a condition which we call left-solvability over X, and which is satisfied by every solvable group G. The importance of an algebraic condition like solvability for problems such as this was apparently first recognized by John von Neumann, in a paper [12] in 1929 in which he studied the existence of finitely additive measures invariant under the action of a group of transformations. Our Theorem 2 can readily be seen to be a generalization of a famous extension theorem of Riesz, to which it reduces when $X_1 = X$ and G consists of the identity alone. It also generalizes a lemma of Parthasarathy and Varadhan [10]. A corollary (in § 2) which analogously generalizes the Hahn-Banach theorem contains the principal result in a paper by R. P. Agnew and A. P. Morse [1], some of the results in a paper by V. L. Klee [5] and a lemma by M. M. Day [3].

The extension theorem is used in § 5 to construct a type of generalized limit for sequences, with larger domain and stronger invariance properties than the familiar Banach limit. In § 6 it is used to