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A NOTE ON PROPOSITION OBSERVABLES

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We consider some questions which were brought up in a previous paper: (1) If the product of two proposition observables is a proposition observable, do the corresponding propositions split? (2) What is the relationship, if any, between the concepts of compatibility and simultaneity of proposition observables? It is shown that the answer to (1) is yes and as a corollary we find the partial answer to (2) that compatibility implies simultaneity. It is also proved that the sum of two proposition observables is a proposition observable if and only if the corresponding propositions are orthogonal and also that $(x_a \circ x_b)^n$ converges weakly to $x_{a \wedge b}$ as $n \to \infty$.

We assume that all observables are defined on a quite full logic satisfying conditions U and E. (See [1] for definitions and notation.) Recall that an observable x is a *proposition observable* if $\sigma(x) \subset \{0, 1\}$. (These observables are also called *questions*, cf. [3].) If $x(\{1\}) = a$ we denote x by x_a. It is clear that proposition observables behave, in some respects, like orthogonal (self-adjoint) projections on a Hilbert space and for this reason it is natural to consider certain properties of projections and ask whether these properties are retained by proposition observables. For example, it is easy to show that if A and B are orthogonal projections then $A + B$ is an orthogonal projection if and only if $AB = BA = 0$ and AB is an orthogonal projection if and only if $AB = BA$ ([4], Th. 13.4). Also if C is the orthogonal projection on the range of A intersected with the range of B then C is the strong operator limit of the sequence $A, BA, ABA, BABA, \cdots$ ([4] Th. 13.7). We show that these results generalize to proposition observables. Recall that the *product* $x \circ y$ of two bounded observables x, y is $x \circ y = \frac{1}{2}[(x + y)^2 - x^2 - y^2]$ and that x and y are *compatible* if $x \circ (z \circ y) = (x \circ z) \circ y = (x \circ y) \circ z$ for all bounded observables z.

2. The theorems.

LEMMA 1. *The following statements are equivalent.* (i) $a \leftrightarrow b$. (ii) $m(a) = 1$, $m(a \wedge b) = 0$ *implies* $m(b) = 0$ *for any state* m. (iii) $\sigma(x_a + x_b) \subset \{0, 1, 2\}$.

Proof. Since $a = (a - a \wedge b) + (a \wedge b)$ and $b = (b - a \wedge b) + (a \wedge b)$, $a \leftrightarrow b$ if and only if $(a - a \wedge b) \perp (b - a \wedge b)$. But the latter holds if and only if $m(a - a \wedge b) = 1$ implies $m(b - a \wedge b) = 0$. This last condition is equivalent to (ii) and hence (i) and (ii) are equivalent.