INTRINSIC TOPOLOGIES IN A TOPOLOGICAL LATTICE

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It is shown that if (L, T) is a compact connected modular topological lattice of finite dimension under a topology T, then the topology T, the interval topology of L, the complete topology of L, and the order topology of L are all the same.

There are a variety of known ways in which a lattice may be given a topology, e.g., Frink's interval topology [8], Birkhoff's order topology [4], and Insel's complete topology [9].

A lattice L is a *topological lattice* if and only if L is a Hausdorff space in which the two lattice operations are continuous.

In this paper we give some of the relationships between topological lattice and its intrinsic topologies and extend a theorem of Dyer and Shields [7] and a result of Anderson [2]. We shall finally prove the main theorem stated above.

We shall use $A \wedge B$ and $A \vee B$ for a pair of subsets A and B of a lattice L to denote the sets $\{a \wedge b \mid a \in A$ and $b \in B\}$ and $\{a \vee b \mid a \in A$ and $b \in B\}$, respectively. For a subset A of L, A^* is the closure of A. The empty set is written as $\square$.

By the *interval topology* of a lattice L, denoted by $I(L)$, we mean the topology defined by taking the closed intervals $\{a \wedge L, a \vee L \mid a \in L\}$ as a sub-base for the closed sets. It is easy to see that if (L, T) is a topological lattice and if $I(L)$ is Hausdorff, then (L, T) is compact if and only if $T = I(L)$ and L is complete.

For a net $\{x_\alpha \mid \alpha \in D\}$ in a complete lattice L, if $\limsup \{x_\alpha \mid \alpha \in D\} = \liminf \{x_\alpha \mid x_\alpha \in D\} = x$, we say that the net $\{x_\alpha\}$ order converges to x. We define a subset M of a complete lattice L to be *closed* in the *order topology* of L, denoted by $O(L)$, if and only if no net in M converges to a point outside of M.

The following two lemmas are immediate:

LEMMA 1. *If (L, T) is a compact topological lattice, and if $\{x_\alpha \mid \alpha \in D\}$ is a monotone decreasing net in L with* $\inf \{x_\alpha \mid \alpha \in D\} = a$, *then the net converges to a in T. The dual argument is also true.*

LEMMA 2. *If (L, T) is a compact topological lattice, then $T \subset O(L)$. Moreover, if $O(L)$ is also compact, then $T = O(L)$.*

By a *complete subset* C of a lattice L we shall mean a nonempty subset C of L such that for each nonempty subset S of C, S possesses both a sup S and an inf S in L, and furthermore, both sup S and