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MEROMORPHIC MINIMAL SURFACES

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Meromorphic minimal surfaces are defined in this paper, and some of their differential-geometric properties are noted. The first fundamental theorem of Nevanlinna for meromorphic functions of a complex variable is extended so as to apply to these surfaces, as is the Ahlfors-Shimizu spherical version of this theorem. For these results, the classical proximity and enumerative functions of complex-variable theory are generalized, and a new visibility function is introduced. Convexity properties of some of these functions are established.

For plane meromorphic maps, the visibility function vanishes at all points on the plane but is positive at all other points of space. In general, in the present development, the sum of the enumerative function and the visibility function corresponds to the enumerative function in the classical theory.

Let a surface S be given by

$$S: x_j = x_j(u, v)\ , \qquad j = 1, 2, 3\ . \tag{1}$$

Then S is said to be given in terms of *isothermal parameters* (u, v) if and only if the representation (1) is such that

$$E = G = \lambda(u, v)\ , \qquad F = 0\ , \tag{2}$$

where

$$E = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \qquad F = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)\left(\frac{\partial x_j}{\partial v}\right), \qquad G = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2. \tag{3}$$

Such an *isothermal representation* is conformal, or angle-preserving, except at points where $\lambda(u, v) = 0$.

According to a theorem of Weierstrass [13, p. 27], a necessary and sufficient condition that a surface S, given in terms of isothermal parameters, be minimal is that the coordinate functions be harmonic, that is, that for all $(u, v) \in D$ the functions $x_j(u, v)$, $j = 1, 2, 3$, satisfy the equation

$$\Delta x_j(u, v) = 0\ , \tag{4}$$

where Δ denotes the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\ . \tag{5}$$

Then in any simply connected part of D, the functions given by (1)