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BOUNDED APPROXIMATION BY RATIONAL FUNCTIONS

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Let D be a bounded open subset of the complex plane $\mathbb{C}$ which is the interior of its closure, and let h be a bounded analytic function on D. The classical theorem of Runge implies that there is a sequence of rational functions with poles in the complement of the closure of D which converges to h uniformly on compact subsets of D. The question naturally arises as whether this sequence may be chosen so that the supremum norms over D of the rational functions remain uniformly bounded. Of course, if the boundary of D consists of a finite number of disjoint circles (that is, D is a circle domain), then it is a classical result that the approximating sequence may be chosen so that their norms do not exceed the norm of h. But suppose that the boundary of D is quite complicated or D has infinitely many components in its complement. This general question has been the subject of several recent papers and is the subject of this one.

In [4] Rubel and Shields showed that if the complement of the closure of D is connected, then there is a sequence $\{h_n\}$ of polynomials with $\| h_n \| \leqq \| h \|$ and $h_n(z) \rightarrow h(z)$ for each z in D. Ahern and Sarason extended this result in [2] and proved that if a bounded open set D is the interior of its closure and has only finitely many components in its complement, then such bounded pointwise approximation is always possible, where the approximating functions have poles in the (finitely many) components of the complement of the closure of D, and their norms on D do not exceed the norm of the limit function.

The chief results in this paper show that rather elementary techniques may be used to extend the theorems of Rubel-Shields and Ahern-Sarason to certain domains with infinitely many complementary components.

We first introduct some notation to be used throughout the remainder of the paper: U is the open unit disc, $\{z \mid |z| < 1\}$; Γ is the unit circle, $\{z \mid |z| = 1\}$; if D is an open set, $H^\infty(D)$ is the space of bounded analytic functions on D and $\bar{D}$ denotes the closure of D; if K is a compact set, then $R(K)$ is the uniform closure on K of the rational functions with poles off K; finally, ∂S denotes the boundary of S.

THEOREM 1. *Let $S_1, S_2, \cdots$ be a sequence of disjoint closed discs in the open unit disc U which are centered on the positive real axis*