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QF-3 RINGS WITH ZERO SINGULAR IDEAL

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Let R be a ring with identity. R is left QF-3 if R has a minimal faithful (left) module, i.e., a faithful (left) module, which is (isomorphic to) a summand of every faithful (left) module. We show that left QF-3 rings are characterized by the existence of a faithful projective-injective left ideal with an essential socle which is a finite sum of simple modules. The main result is a structure theorem for left and right QF-3 rings with zero left singular ideal. This theorem gives several descriptions of this class of rings. Among these is that the above rings are exactly the orders (containing units) with essential left and right socles in semi-simple two-sided (complete) quotient rings.

Let R be a ring with identity. R is *left* QF-3 if R has a minimal faithful (left) module, i.e., a faithful (left) module which is (isomorphic to) a summand of every faithful (left) module. Finite dimensional algebras with this property were introduced by R. M. Thrall as a generalization of quasi-Frobenius algebras, and other authors have considered Artinian and semi-primary QF-3 rings. For a semi-primary ring, being left QF-3 is equivalent to the existence of a faithful projective-injective left ideal. The first theorem shows that to characterize left QF-3 rings in general one must assume that some faithful projective-injective left ideal has an essential socle which is a finite sum of simple modules. The rest of this paper concerns rings with zero singular ideal which are either QF-3 or have faithful projective-injective one-sided ideals. The main result is a structure theorem for left and right QF-3 rings with zero left singular ideal. This theorem gives several descriptions of this class of rings. Among these is that the above rings are exactly the orders (containing units) with essential left and right socles in semi-simple[1] two-sided (complete) quotient rings. This theorem extends and unifies result of M. Harada [5, 6], J. P. Jans [8], and H. Mochizuki [12].

RESULTS. A submodule N of a module M is *essential* in M if every nonzero submodule of M meets N nontrivially. The *singular submodule* $Z(M) = \{x \in M \mid Ix = 0$ for some essential left ideal I of $R\}$. $Z({}_RR)$ is an ideal of R called the left singular ideal of R. An R-module M is called *uniform* if every nonzero submodule of M is essential in M. If M and N are R-modules with M uniform and

[1] Throughout this paper, *semi-simple* means *semi-simple Artinian*.